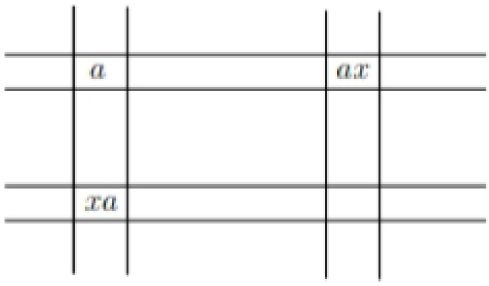

FIGURE 5. The egg box diagram of D_a.

(3) If S is a band then S is regular as $e = e^3$ for all $e \in S$; S need not be inverse.
(4) B is regular because $(a, b) = (a, b)(b, a)(a, b)$ for all $(a, b) \in B$. Furthermore, B is inverse - see later.
(5) $\mathcal{M}^0$ is regular (see "Proposition 7.3, Rees Matrix Facts").
(6) $\mathcal{T}_X$ is regular (see Exercises).
(7) $(\mathbb{N}, +)$ is not regular as, for example $1 \neq 1 + a + 1$ for any $a \in \mathbb{N}$.

Theorem 8.6. *[Inverse Semigroup Theorem] A semigroup S is inverse iff S is regular and $E(S)$ is a semilattice (i.e. $ef = fe$ for all $e, f \in E(S)$).*

Proof. ($\Leftarrow$) Let $a \in S$. As S is regular, a has an inverse by Lemma 8.3. Suppose $x, y \in V(a)$. Then

$$a \underset{(1)}{=} axa \qquad x \underset{(2)}{=} xax \qquad a \underset{(3)}{=} aya \qquad y \underset{(4)}{=} yay,$$

so $ax, xa, ay, ya \in E(S)$. This gives us that

$$\begin{aligned} x &\underset{(2)}{=} xax \underset{(3)}{=} x(aya)x = (xa)(ya)x = (ya)(xa)x = y(axa)x \\ &\underset{(1)}{=} yax \underset{(3)}{=} y(aya)x = y(ay)(ax) = y(ax)(ay) = y(axa)y \underset{(1)}{=} yay \underset{(4)}{=} y. \end{aligned}$$

So $|V(a)| = 1$ and S is inverse.

Conversely, suppose S is inverse. Let a' denote the *unique* inverse of $a \in S$. Certainly S is regular. Let $e \in E(S)$. Then e is an inverse of e, because $e = eee$ and $e = eee$, so the inverse of any idempotent e is just itself: $e' = e$.

Let $e, f \in E(S)$. Let $x = (ef)'$. Consider the element fxe. Then

$$(fxe)^2 = (fxe)(fxe) = f(xefx)e = fxe$$

as $x = (ef)'$. So $fxe \in E(S)$ and therefore $fxe = (fxe)'$.