Examples of regular semigroups: any band, Rees matrix semigroups, groups.
Examples of non-regular semigroups: $(\mathbb{N}, +), (\mathbb{Z}, *)$
Nontrivial null (or zero) semigroups i.e. $S = X \cup \{0\}$ with $X \neq \emptyset$ and all products are 0.
Note that $(\mathbb{N}, +)$ has *no* regular element.

DEFINITION 8.2. An element $a' \in S$ is an *inverse* of a if

$$a = aa'a \text{ and } a' = a'aa'.$$

We denote by $V(a)$ the set of inverses of a.

If G is a group then $V(a) = \{a^{-1}\}$ for all $a \in G$.

CAUTION: Inverses need not be unique. For example, in a rectangular band $T = I \times \Lambda$,

$$(i, j)(k, \ell)(i, j) = (i, j)$$
$$(k, \ell)(i, j)(k, \ell) = (k, \ell)$$

for any (i, j) and (k, ℓ). So every element is an inverse of every other element.

Lemma 8.3. *If $a \in S$, then a is regular $\Leftrightarrow V(a) \neq \emptyset$.*

Proof. If $V(a) \neq \emptyset$, clearly a is regular. Conversely suppose that a is regular. Then there exists $x \in S$ with $a = axa$. Put $a' = xax$. Then

$$aa'a = a(xax)a = (axa)xa = axa = a,$$

and

$$a'aa' = (xax)a(xax) = x(axa)(xax)$$
$$= xa(xax) = x(axa)x = xax = a'.$$

So $a' \in V(a)$. □

NOTE. If $a = axa$ then

$$(ax)^2 = (ax)(ax) = (axa)x = ax$$

so $ax \in E(S)$ and dually, $xa \in E(S)$. Moreover

$$a = axa \quad ax = ax \Rightarrow a \,\mathcal{R}\, ax,$$
$$a = axa \quad xa = xa \Rightarrow a \,\mathcal{L}\, xa.$$

DEFINITION 8.4. S is *inverse* if $|V(a)| = 1$ for all $a \in S$, i.e. every element has a unique inverse.

EXAMPLE 8.5.

(1) Groups are inverse; $V(a) = \{a^{-1}\}$.
(2) A rectangular band T is regular; but (as every element of T is an inverse of every other element) T is not inverse (unless T is trivial).