(iv) *the $\mathcal{J}$-classes are* $\{0\}$ *and* $S \setminus \{0\}$.

Proof. (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is a standard exercise.

(ii) $\Rightarrow$ (iii): Let $a \in S \setminus \{0\}$. Then

$$S = SaS \subseteq S^1 a S^1 \subseteq S$$

and therefore $S = S^1 a S^1$.

(i) $\Rightarrow$ (ii): Since $S^2 \neq 0$ and S^2 is an ideal, then $S^2 = S$. Therefore

$$S^3 = SS^2 = S^2 = S \neq 0.$$

Let $I = \{x \in S \mid SxS = 0\}$. Clearly $0 \in I$ and hence $I \neq \emptyset$. If $x \in I$ and $s \in S$, then

$$0 \subseteq SxsS \subseteq SxS = 0.$$

Therefore $SxsS = 0$ and so $xs \in I$. Dually $sx \in I$; therefore I is an ideal. If $I = S$, then

$$\begin{aligned} S^3 &= SIS, \\ &= \bigcup_{x \in I} SxS, \\ &= 0. \end{aligned}$$

This is a contradiction, therefore $I \neq S$. Hence $I = 0$. Let $a \in S \setminus \{0\}$. Then SaS is an ideal and as $a \notin I$ we have $SaS \neq 0$. Hence $SaS = S$. □

Corollary 7.16. *Let S be completely 0-simple. Then S contains a non-zero idempotent.*

Proof. Let $a \in S \setminus \{0\}$. Then $SaS = S$, therefore there exists a $u, v \in S$ with $a = uav$. So,

$$a = uav = u^2 a v^2 = \cdots = u^n a v^n$$

for all n. Hence $u^n \neq 0$ for all $n \in \mathbb{N}$. Pick n, m with $u^n \mathrel{\mathcal{R}} u^{n+1}$, $u^m \mathrel{\mathcal{L}} u^{m+1}$. Notice

$$u^{n+1} \mathrel{\mathcal{R}} u^{n+2}$$

as $\mathcal{R}$ is a left congruence. Similarly,

$$u^{n+2} \mathrel{\mathcal{R}} u^{n+3}$$

we deduce that $u^n \mathrel{\mathcal{R}} u^{n+t}$ for all $t \geqslant 0$. Similarly $u^m \mathrel{\mathcal{L}} u^{m+t}$ for all $t \geqslant 0$. Let $s = \max\{m, n\}$. Then $u^s \mathrel{\mathcal{R}} u^{2s}$, $u^s \mathrel{\mathcal{L}} u^{2s}$ so $u^s \mathrel{\mathcal{H}} u^{2s} = (u^s)^2$. Hence by Corollary 5.7, u^s lies in a subgroup. Therefore $u^s \mathrel{\mathcal{H}} e$ for some idempotent e. As $u^s \neq 0$ and $H_0 = \{0\}$, we have $e \neq 0$. □

Theorem 7.17 (Rees' Theorem - 1941)**.** *Let S be a semigroup with zero. Then S is completely 0-simple* $\Leftrightarrow$ *S is isomorphic to a Rees Matrix Semigroup over a group.*