for all $n \in \mathbb{N}$. By $(\star)$, there exists n with $(xu)^n \mathcal{L} (xu)^{n+1}$. Therefore

$$b = (xu)^n b(vy)^n \mathcal{L} (xu)^{n+1} b(vy)^n = xu\big((xu)^n b(vy)^n\big) = xub.$$

Therefore $b \mathcal{L} xub$, so

$$S^1 b = S^1 xub \subseteq S^1 ub \subseteq S^1 b.$$

So $S^1 b = S^1 ub$, which means that $b \mathcal{L} ub$. Dually, $b \mathcal{R} bv$. Therefore $a = ubv \mathcal{R} ub \mathcal{L} b$. So $a \mathcal{D} b$ and $\mathcal{J} \subseteq \mathcal{D}$. Consequently, $\mathcal{D} = \mathcal{J}$. □

As a consequence we have the following:

Corollary 7.13. *If a semigroup S has M_L and M_R, then it satisfies $(\star)$ and thus $\mathcal{D} = \mathcal{J}$.*

In the same vein we have:

Lemma 7.14. The Rectangular Property:
Let S satisfy $(\star)$. Then for all $a, b \in S$ we have

(i) $a \mathcal{J} ab \Leftrightarrow a \mathcal{D} ab \Leftrightarrow a \mathcal{R} ab$,
(ii) $b \mathcal{J} ab \Leftrightarrow b \mathcal{D} ab \Leftrightarrow b \mathcal{L} ab$.

Proof. We prove (i), (ii) being dual. Now,

$$a \mathcal{J} ab \Leftrightarrow a \mathcal{D} ab$$

as $\mathcal{D} = \mathcal{J}$. Clearly if $a \mathcal{R} ab$ then $a \mathcal{D} ab$; as $\mathcal{R} \subseteq \mathcal{D}$.
Conversely, If $a \mathcal{J} ab$ then there exists $x, y \in S^1$ with

$$a = xaby = xa(by) = x^n a(by)^n$$

for all n. Pick n with $(by)^n \mathcal{R} (by)^{n+1}$. Then

$$a = x^n a(by)^n \mathcal{R} x^n a(by)^{n+1} = x^n a(by)^n by = aby.$$

Now

$$aS^1 = abyS^1 \subseteq abS^1 \subseteq aS^1.$$

Hence $aS^1 = abS^1$ and $a \mathcal{R} ab$. □

7.1. Completely 0-simple semigroups

Let S have a 0. Recall that S is *0-simple* if and only if 0 (properly, $\{0\}$) and S are the only ideals and $S^2 \neq 0$. If in addition S has M_R and M_L, then S is *completely 0-simple*.

Lemma 7.15. *[0-Simple Lemma] Let S have a 0 and $S^2 \neq 0$. Then the following are equivalent:*

(i) *S is 0-simple,*
(ii) *$SaS = S$ for all $a \in S \setminus \{0\}$,*
(iii) *$S^1 aS^1 = S$ for all $a \in S \setminus \{0\}$,*