However, since $0 < 1 < 2 < \dots$ we have

$$B(0,0) \supset B(1,1) \supset B(2,2) \supset \dots$$

so there exists infinite descending chains. Hence B **does not have** M_L or M_R.

EXAMPLE 7.9. Let $\mathcal{M}^0 = \mathcal{M}^0(G; I; \Lambda; P)$ be a Rees Matrix Semigroup over a group G. Then $\mathcal{M}^0$ has M_L, M_R, M^L and M^R.

Proof. We show that the length of the strict chains is at most 2. Suppose $\alpha\mathcal{M}^0 \subsetneq \beta\mathcal{M}^0$. We could have $\alpha = 0$. If $\alpha \neq 0$ then $\alpha\mathcal{M}^0 \neq \{0\}$ so $\beta \neq 0$ and we have $\alpha = (i, g, \lambda)$, $\beta = (j, h, \mu)$ and $\alpha = \beta\gamma$ for some $\gamma = (\ell, k, \nu)$. Then

$$(i, g, \lambda) = (j, h, \mu)(\ell, k, \nu) = (j, h\rho_{\mu\ell}k, \nu).$$

This gives us that $i = j$ and so $\alpha \mathcal{R} \beta$ and $\alpha\mathcal{M}^0 = \beta\mathcal{M}^0$.
Summarising, $0\mathcal{M}^0 \subset \alpha\mathcal{M}^0$ for all non-zero α. But if $\alpha \neq 0$ and $\alpha\mathcal{M}^0 \subsetneq \beta\mathcal{M}^0$, then $\alpha\mathcal{M}^0 = \beta\mathcal{M}^0$. Hence $\mathcal{M}^0$ has M_R and M^R; dually $\mathcal{M}^0$ has M_L and M^L. □

DEFINITION 7.10. A 0-simple semigroup is *completely 0-simple* if it has M_R and M_L.

By above, any Rees Matrix Semigroup over a group is completely 0-simple. Our aim is to show that every completely 0-simple semigroup is isomorphic to a Rees Matrix Semigroup over a group.

Theorem 7.11 (The $\mathcal{D} = \mathcal{J}$ Theorem). *Suppose*

$$(\star) \qquad \begin{cases} \forall\, a \in S,\ \exists\, n \in \mathbb{N} \text{ with } a^n \ \mathcal{L}\ a^{n+1}, \\ \forall\, a \in S,\ \exists\, m \in \mathbb{N} \text{ with } a^m \ \mathcal{R}\ a^{m+1}. \end{cases}$$

Then $\mathcal{D} = \mathcal{J}$.

EXAMPLE 7.12.

(1) If S is a band, $a = a^2$ for all $a \in S$ and so $(\star)$ holds.
(2) Let S be a semigroup having M_L and let $a \in S$. Then

$$S^1a \supseteq S^1a^2 \supseteq S^1a^3 \supseteq \dots.$$

Since S has M_L, we have that this sequence stabilizes, so there exists $n \in \mathbb{N}$ such that $S^1a^n = S^1a^{n+1}$ which means that $a^n \ \mathcal{L}\ a^{n+1}$. Similarly, if S has M_R, then for every $a \in S$ there exists $m \in \mathbb{N}$ such that $a^m \ \mathcal{R}\ a^{m+1}$.

Proof. of $\mathcal{D} = \mathcal{J}$ Theorem
We know $\mathcal{D} \subseteq \mathcal{J}$. Let $a, b \in S$ with $a \ \mathcal{J}\ b$. Then there exists $x, y, u, v \in S^1$ with

$$b = xay, \ a = ubv.$$

Then

$$b = xay = x(ubv)y = (xu)b(vy) = (xu)^2b(vy)^2 = \dots = (xu)^nb(vy)^n$$