Proof. If every chain with $\supseteq$ terminates, then clearly we cannot have an infinite strict chain

$$S^1a_1 \supset S^1a_2 \supset \dots$$

So S has M_L.
Conversely, suppose S has M_L and we have a chain

$$S^1a_1 \supseteq S^1a_2 \supseteq \dots$$

Let the strict inclusions be at the j_ith steps:

$$\begin{aligned} S^1a_1 = S^1a_2 = \dots = S^1a_{j_1} \supset S^1a_{j_1+1} = S^1a_{j_1+2} \\ = \dots = S^1a_{j_2} \supset S^1a_{j_2+1} = \dots \end{aligned}$$

Then

$$S^1a_{j_1} \supset S^1a_{j_2} \supset \dots$$

As S has M_L, this chain is finite with length n say. Then

$$S^1a_{j_n+1} = S^1a_{j_n+2} = \dots$$

and our sequence has stabilised. □

DEFINITION 7.6. The *ascending chain condition* (a.c.c.) on principal ideals on left/right ideals M^L (M^R) is defined as above but with the inclusions reversed.

The analogue of the Chain Lemma holds for M^L and (M^R).

EXAMPLE 7.7. Every finite semigroup has M_L, M_R, M^L, M^R. For example, if

$$S^1a_1 \supset S^1a_2 \supset S^1a_3 \supset \dots,$$

then in every step, the cardinality of the sets must decrease at least by one, so the length of a strict sequence cannot be greater than $|S|$.

EXAMPLE 7.8. The Bicyclic semigroup B has M^L and M^R. We know

$$B(x, y) = \{(p, q) \mid q \geqslant y\}$$

and so

$$B(x, y) \subseteq B(u, v) \Leftrightarrow y \geqslant v,$$

and inclusion is strict if and only if $y > v$. If we had an infinite chain

$$B(x_1, y_1) \subset B(x_2, y_2) \subset B(x_3, y_3) \subset \dots$$

then we would have

$$y_1 > y_2 > y_3 > \dots,$$

which is impossible in $\mathbb{N}$.
Hence M^L holds, dually M^R holds.