(6) $\{0\}$ is a $\mathcal{D}$-class and a $\mathcal{J}$-class. If $(i, a, \lambda), (j, b, \mu) \in \mathcal{M}^0$ then

$$(i, a, \lambda) \; \mathcal{R} \; (i, a, \mu) \; \mathcal{L} \; (j, b, \mu)$$

so $(i, a, \lambda) \; \mathcal{D} \; (j, b, \mu)$ and so $(i, a, \lambda) \; \mathcal{J} \; (j, b, \mu)$. Therefore $\mathcal{D} = \mathcal{J}$ and $\{0\}$ and $\mathcal{M}^0 \setminus \{0\}$ are the only classes.

(7) We have already shown that the only $\mathcal{J}$-classes are $\{0\}$ and $\mathcal{M}^0 \setminus \{0\}$. Let $i \in I$, then there exists $\lambda \in \Lambda$ with $p_{\lambda i} \neq 0$ so $(i, 1, \lambda)^2 \neq 0$. Therefore $(\mathcal{M}^0)^2 \neq 0$ and so $\mathcal{M}^0$ is 0-simple.

(8) If $xy \; \mathcal{R} \; x$, then clearly $xy \; \mathcal{D} \; x$, because $\mathcal{R} \subseteq \mathcal{D}$. For the other direction, suppose that $xy \; \mathcal{D} \; x$. Notice that the two $\mathcal{D}$-classes are zero and everything else. If $xy = 0$, then necessarily $x = 0$, because $D_0 = \{0\}$. If $xy \neq 0$, then necessarily $x, y \neq 0$, so we have that

$$x = (i, a, \lambda) \qquad y = (j, b, \mu).$$

Then $xy = (i, ap_{\lambda j}b, \mu)$, so $xy \; \mathcal{R} \; x$. The result for $\mathcal{L}$ is dual. □

Some more facts!

(9) Put $H_{i\lambda} = \big\{(i, a, \lambda) \mid a \in G\big\}$. By (5) we have $H_{i\lambda}$ is an $\mathcal{H}$-class ($H_{i\lambda} = H_{(i,e,\lambda)}$). If $p_{\lambda i} \neq 0$ we know $(i, p_{\lambda i}^{-1}, \lambda)$ is an idempotent and so $H_{i\lambda}$ is a group, by the Maximal Subgroup Theorem. The identity is $(i, p_{\lambda i}^{-1}, \lambda)$ and $(i, a, \lambda)^{-1} = (i, p_{\lambda i}^{-1} a^{-1}, p_{\lambda i}^{-1}, \lambda)$.

(10) If $p_{\lambda i} \neq 0$ and $p_{\mu j} \neq 0$ then $H_{i\lambda} \simeq H_{j\mu}$. It is clear that $(i, a, \lambda) \mapsto (j, a, \mu)$ is a bijection, but this is not in general a morphism. *Exercise: find a morphism!*

Chain conditions

A finitary property is a property held by all finite semigroups: chain conditions are one kind of finitary property.

Definition 7.4. A semigroup S has M_L if there are no infinite chains

$$S^1a_1 \supset S^1a_2 \supset S^1a_3 \supset \dots$$

of principal left ideals. M_L is the *descending chain condition* (d.c.c.) on principal left ideals.
The left/right dual is M_R.

Lemma 7.5 (The Chain Lemma). *The semigroup S has M_L if and only if any chain*

$$S^1a_1 \supseteq S^1a_2 \supseteq \dots$$

terminates (stabilizes) i.e. there exists $n \in \mathbb{N}$ with

$$S^1a_n = S^1a_{n+1} = \dots$$