Definition 7.2. $a \in S$ is *regular* if there exists $x \in S$ with

$$a = axa.$$

S is *regular* if every $a \in S$ is regular.

If S is regular then $a \,\mathcal{R}\, b \Leftrightarrow aS = bS \Leftrightarrow$ there exists $s, t \in S$ with $a = bs$ and $b = at$, etc.

Proposition 7.3. Rees matrix facts *Let $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$ be a Rees Matrix Semigroup over a group G.*

(1) (i, a, λ) *is idempotent* $\Leftrightarrow p_{\lambda i} \neq 0$ *and* $a = p_{\lambda i}^{-1}$.
(2) $\mathcal{M}^0$ *is regular.*
(3) $(i, a, \lambda) \,\mathcal{R}\, (j, b, \mu) \Leftrightarrow i = j$.
(4) $(i, a, \lambda) \,\mathcal{L}\, (j, b, \mu) \Leftrightarrow \lambda = \mu$.
(5) $(i, a, \lambda) \,\mathcal{H}\, (j, b, \mu) \Leftrightarrow i = j$ *and* $\lambda = \mu$.
(6) *The* $\mathcal{D} = \mathcal{J}$*-classes are* $\{0\}$ *and* $\mathcal{M}^0 \setminus \{0\}$ *(so* 0 *and* $\mathcal{M}^0$ *are the only ideals).*
(7) $\mathcal{M}^0$ *is 0-simple.*
(8) *The so-called rectangular property:*

$$\left.\begin{array}{l} xy \,\mathcal{D}\, x \Leftrightarrow xy \,\mathcal{R}\, x \\ xy \,\mathcal{D}\, y \Leftrightarrow xy \,\mathcal{L}\, y \end{array}\right\} \forall\, x, y \in \mathcal{M}^0$$

Proof. (1) We have that

$$\begin{aligned}(i, a, \lambda) \in E(\mathcal{M}^0) &\Leftrightarrow (i, a, \lambda) = (i, a, \lambda)(i, a, \lambda),\\ &\Leftrightarrow p_{\lambda i} \neq 0, (i, a, \lambda) = (i, ap_{\lambda i}a, \lambda),\\ &\Leftrightarrow p_{\lambda i} \neq 0, a = ap_{\lambda i}a,\\ &\Leftrightarrow p_{\lambda i} \neq 0 \text{ and } p_{\lambda i} = a^{-1}.\end{aligned}$$

(2) $0 = 000$ so 0 is regular. Let $(i, a, \lambda) \in \mathcal{M}^0 \setminus \{0\}$ then there exists $j \in I$ with $p_{\lambda j} \neq 0$ and there exists $\mu \in \Lambda$ with $p_{\mu i} \neq 0$. Now,

$$(i, a, \lambda)(j, p_{\lambda j}^{-1} a^{-1} p_{\mu i}^{-1}, \mu)(i, a, \lambda) = (i, a, \lambda)$$

and hence $\mathcal{M}^0$ is regular.

(3) $\{0\}$ is an $\mathcal{R}$-class. If $(i, a, \lambda) \,\mathcal{R}\, (j, b, \mu)$ then there exists $(k, c, \nu) \in \mathcal{M}^0$ with

$$(i, a, \lambda) = (j, b, \mu)(k, c, \nu) = (j, bp_{\mu k}c, \nu)$$

and so $i = j$. Conversely, if $i = j$, pick k with $p_{\mu k} \neq 0$. Then

$$(i, a, \lambda) = (j, b, \mu)(k, p_{\mu k}^{-1} b^{-1} a, \lambda)$$

and together with the dual we have $(i, a, \lambda) \,\mathcal{R}\, (j, b, \mu)$

(4) Dual.
(5) This comes from (3) and (4) above.