Proof. We prove that if $H^2 \cap H \neq \emptyset$, then H is a subgroup. This is exactly the statement of the theorem.
So suppose $H^2 \cap H \neq \emptyset$. Then there exists $a, b, c \in H$ such that $ab = c$. Since $a \mathcal{R} c$, $\rho_b : H_a \to H_c$ is a bijection. But $H_a = H_c = H$ so $\rho_b : H \to H$ is a bijection. Hence $Hb = H$. Dually, $aH = H$.
Let $u, v \in H$. Then $av \in H$ so that as above, $Hv = H$. But then $uv \in H$ and H is a subsemigroup. Further, $vH = H$ so that by a standard argument (see Exercises 1), H is a subgroup of S.

Alternatively Since $b \in H$, $b = db$ for some $d \in H$. As $b \mathcal{R} d$, $d = bs$ for some $s \in S^1$ and then $d = bs = dbs = d^2$. Hence H contains an idempotent, so (by the Maximal Subgroup Theorem) it is a subgroup. □

Corollary 6.10. *$a \mathcal{H} a^2 \Leftrightarrow H_a$ is a subgroup.*

Proof. We know H_a is a subgroup $\Rightarrow a, a^2 \in H_a$ so $a \mathcal{H} a^2$.
Conversely, if $a\mathcal{H}a^2$, then $a^2 \in H_a \cap (H_a)^2$. Hence $H_a \cap (H_a)^2 \neq \emptyset$. So, by Green's Lemma, H_a is a subgroup. □

7. Rees Matrix Semigroups

Just as the main building blocks of groups are simple groups, the main building blocks of semigroups are 0-simple semigroups.
In general, the structure of 0-simple semigroups is very complicated. In the finite case and, more generally, in case certain *chain conditions* hold, their structure is transparent - they can be described by a group and a matrix.

Construction: Let G be a group, let I, Λ be non-empty sets and let P be a $\Lambda \times I$ matrix over $G \cup \{0\}$ such that every row and every column of P contains at least one non-zero entry.

$\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$ is the set

$$I \times G \times \Lambda \cup \{0\}$$

with binary operation given by $0n = 0 = n0$ for all $n \in \mathcal{M}^0$ and

$$(i, a, \lambda)(k, b, \mu) = \begin{cases} 0 & \text{if } p_{\lambda k} = 0, \\ (i, a p_{\lambda k} b, \mu) & \text{if } p_{\lambda k} \neq 0. \end{cases}$$

Check that $\mathcal{M}^0(G; I, \Lambda; P)$ is a semigroup with zero 0.

Definition 7.1. $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$ is called a *Rees Matrix Semigroup over*