Then $\rho_s : L_a \to L_b$ and $\rho_{s'} : L_b \to L_a$ are mutually inverse, $\mathcal{R}$-class preserving bijections (i.e. if $c \in L_a$, then $c \,\mathcal{R}\, c\rho_s$ and if $d \in L_b$ then $d \,\mathcal{R}\, d\rho_{s'}$).

Proof. If $c \in L_a$ then

$$c\rho_s = cs \;\mathcal{L}\; as = b,$$

because $\mathcal{L}$ is a right congruence. So $c\rho_s \,\mathcal{L}\, b$ therefore $\rho_s : L_a \to L_b$. Dually $\rho_{s'} : L_b \to L_a$.

Let $c \in L_a$. Then $c = ta$ for some $t \in S$. Now

$$c\rho_s\rho_{s'} = tas\rho_{s'} = tass' = tbs' = ta = c.$$

So $\rho_s\rho_{s'} = I_{L_a}$, dually, $\rho_{s'}\rho_s = I_{L_b}$.

Again, let $c \in L_a$. Then

$$cs = c \cdot s,$$
$$c = cs \cdot s'.$$

Therefore $c \,\mathcal{R}\, cs = c\rho_s$. □

Continuing Lemma 6.6. For any $c \in L_a$ we have $\rho_s : H_c \to H_{cs}$ is a bijection with inverse $\rho_{s'} : H_{cs} \to H_c$. In particular – put $c = a$ then

$$\rho_s : H_a \to H_b \qquad \text{and} \qquad \rho_s : H_b \to H_a$$

are mutually inverse bijections.

Let $s \in S^1$. Then we define $\lambda_s : S \to S$ by $a\lambda_s = sa$.

Lemma 6.7 (Dual of Green's Lemma). *Let $a, b \in S$ be such that $a \,\mathcal{L}\, b$ and let $t, t' \in S$ be such that $ta = b$ and $t'b = a$. Then $\lambda_t : R_a \to R_b$ and $\lambda_{t'} : R_b \to R_a$ are mutually inverse $\mathcal{L}$-class preserving bijections. In particular, for any $c \in R_a$ we have $\lambda_t : H_c \to H_{tc}$, $\lambda_{t'} : H_{tc} \to H_c$ are mutually inverse bijections. So, if $c = a$ we have $\lambda_t : H_a \to H_b$, $\lambda_{t'} : H_b \to H_a$ are mutually inverse bijections.*

Corollary 6.8. *If $a \,\mathcal{D}\, b$ then there exists a bijection $H_a \to H_b$.*

Proof. If $a \,\mathcal{D}\, b$ then there exists $h \in S$ with $a \,\mathcal{R}\, h \,\mathcal{L}\, b$. There exists a bijection $H_a \to H_h$ by Green's Lemma and we also have that there exists a bijection $H_h \to H_b$ by the Dual of Green's Lemma. Therefore there exists a bijection $H_a \to H_b$. □

Thus any two $\mathcal{H}$-classes in the same $\mathcal{D}$-class have the same cardinality (just like any two $\mathcal{R}$- and $\mathcal{L}$-classes).

Theorem 6.9 (Green's Theorem – Strong Version). *Let H be an $\mathcal{H}$-class of S. Then either $H^2 \cap H = \emptyset$ or H is a subgroup of S.*