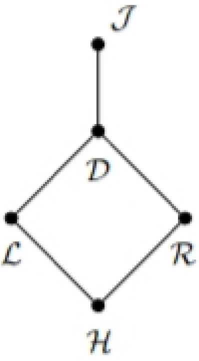
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \subseteq \mathcal{D},$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \subseteq \mathcal{R} \subseteq \mathcal{D}.$$

As $\mathcal{J}$ is an equivalence relation and $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{J}$ we must have $\mathcal{D} \subseteq \mathcal{J}$. This has Hasse Diagram



NOTATION: D_a is the $\mathcal{D}$ class of $a \in S$ and J_a is the $\mathcal{J}$-class of $a \in S$.

NOTE. $H_a \subseteq L_a \subseteq D_a \subseteq J_a$ and also $H_a \subseteq R_a \subseteq D_a \subseteq J_a$.

Egg-Box Pictures

Let D be a $\mathcal{D}$-class. Then for any $a \in D$ we have $R_a \subseteq D = D_a$, and $L_a \subseteq D$. We denote the $\mathcal{R}$-classes as rows and the $\mathcal{L}$-classes as columns. The cells (if non-empty) will be $\mathcal{H}$-classes - we show they are all non-empty!
Let $u, v \in D$ then $u \mathrel{\mathcal{D}} v$. This implies that there exists $h \in S$ with $u \mathrel{\mathcal{R}} h \mathrel{\mathcal{L}} v$, so $R_u \cap L_v \neq \emptyset$, that is, **no cell is empty**. Moreover

$$R_u \cap L_v = R_h \cap L_h = H_h.$$

As $\mathcal{D}$ is an equivalence, S is the union of such "egg-boxes": the rows represent the $\mathcal{R}$-classes, and the columns represent the $\mathcal{L}$-classes.

	u	h	
		v	

6.2. Structure of $\mathcal{D}$-classes

Let S be a semigroup, $s \in S^1$. We define $\rho_s : S \to S$ by $a\rho_s = as$ for all $a \in S$

Lemma 6.6 (Green's Lemma). *Let $a, b \in S$ be such that $a \mathrel{\mathcal{R}} b$ and let $s, s' \in S$ be such that*

$$as = b \qquad \textit{and} \qquad bs' = a.$$