- Suppose that $a\ \nu\ b \nu\ c$ then there exists $x, y \in A$ with

  $$a\ \rho\ x\ \lambda\ b\ \lambda\ y\ \rho\ c.$$

  (Note that first we use that $\nu = \rho \circ \lambda$, and next we use that $\nu = \lambda \circ \rho$.)
  From $x\ \lambda\ b\ \lambda\ y$ we have $x\ \lambda\ y$, so

  $$a\ \rho\ x\ \lambda\ y\ \rho\ c.$$

  Therefore $x\ \nu\ c$ hence there exists $z \in A$ such that $x\ \rho\ z\ \lambda\ c$, therefore $a\ \rho\ z\ \lambda\ c$ and hence $a\ \nu\ c$. Therefore ν is transitive.

We have shown that ν is an equivalence relation. If $(a, b) \in \rho$ then $a\ \rho\ b\ \lambda\ b$ so $(a, b) \in \nu$. Similarly if $(a, b) \in \lambda$ then $a\ \rho\ a\ \lambda\ b$ so $(a, b) \in \nu$. Hence $\rho \cup \lambda \subseteq \nu$.

Now, suppose $\rho \cup \lambda \subseteq \tau$ where τ is an equivalence relation. Let $(a, b) \in \nu$. Then we have $a\ \rho\ c\ \lambda\ b$ for some c. Hence $a\ \tau\ c\ \tau\ b$ so $a\ \tau\ b$ as τ is transitive. Therefore $\nu \subseteq \tau$. □

The smallest equivalence relation containing any ρ and λ is denoted by $\rho \vee \lambda$; we have shown that if ρ and λ commute, then $\rho \vee \lambda = \rho \circ \lambda$.

Definition 6.4. $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, i.e. $a\ \mathcal{D}\ b \Leftrightarrow \exists c \in S$ with $a\ \mathcal{R}\ c\ \mathcal{L}\ b$.

Lemma 6.5 (The $\mathcal{D}$ Lemma). $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$

Proof. We prove that $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$, the proof of the other direction being dual. Suppose that $a\ \mathcal{R} \circ \mathcal{L}\ b$. Then there exists $c \in S$ with

$$a\ \mathcal{R}\ c\ \mathcal{L}\ b$$

There exists $u, v, s, t \in S^1$ with

$$a \underset{(1)}{=} cu \qquad c \underset{(2)}{=} av \qquad c \underset{(3)}{=} sb \qquad b \underset{(4)}{=} tc.$$

Put $d = bu$ then we have

$$a \underset{(1)}{=} cu \underset{(3)}{=} sbu = sd,$$
$$d = bu \underset{(4)}{=} tcu \underset{(1)}{=} ta.$$

Therefore $a\ \mathcal{L}\ d$. Also

$$b \underset{(4)}{=} tc \underset{(2)}{=} tav \underset{(1)}{=} tcuv \underset{(4)}{=} buv = dv.$$

Therefore $b\ \mathcal{R}\ d$ and hence $a\ \mathcal{L} \circ \mathcal{R}\ b$. □

Hence $\mathcal{D}$ is an equivalence relation and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.
By definition

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