6. $\mathcal{D}, \mathcal{J}$ and Green's Lemmas

Recall $S^1aS^1 = \{xay \mid x, y \in S^1\}$.

Definition 6.1. We say that $a \mathcal{J} b$ if and only if

$$a \mathcal{J} b \Leftrightarrow S^1aS^1 = S^1bS^1$$

Check:

$$a \mathcal{J} b \Leftrightarrow \exists\, s, t, u, v \in S^1 \text{ with } a = sbt \quad b = uav.$$

Note. If $a \mathcal{L} b$, then $S^1a = S^1b$ so $S^1aS^1 = S^1bS^1$ so $a \mathcal{J} b$, i.e. $\mathcal{L} \subseteq \mathcal{J}$, dually $\mathcal{R} \subseteq \mathcal{J}$.

Recall: S is *simple* if S is the only ideal of S. If S is simple and $a, b \in S$ then

$$S^1aS^1 = S = S^1bS^1 \qquad \text{so } a \mathcal{J} b$$

and $\mathcal{J} = \omega$ (the universal relation). Conversely if $\mathcal{J} = \omega$ and I is an ideal of S, then pick any $a \in I$ and any $s \in S$. We have

$$s \in S^1sS^1 = S^1aS^1 \subseteq I.$$

Therefore $I = S$ and S is simple.

We have shown that that

$$S \text{ is simple } \Leftrightarrow \mathcal{J} = \omega.$$

Similarly if S has a zero, then $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes iff $\{0\}$ and S are the only ideals.

6.1. Composition of Relations

Definition 6.2. If ρ and λ are relations on A we define

$$\rho \circ \lambda = \big\{(x, y) \in A \times A \mid \exists\, z \in A \text{ with } (x, z) \in \rho \text{ and } (z, y) \in \lambda\big\}.$$

Lemma 6.3. *If ρ, λ are equivalence relations and if $\rho \circ \lambda = \lambda \circ \rho$ then $\rho \circ \lambda$ is an equivalence relation. Also, it is the* smallest *equivalence relation containing $\rho \cup \lambda$.*

Proof. Put $\nu = \rho \circ \lambda = \lambda \circ \rho$

- for any $a \in A$, $a \,\rho\, a \,\lambda\, a$ so $a \,\nu\, a$ and ν is reflexive.
- Symmetric - an exercise.