Let $e, f \in E(S)$ with $e \neq f$. Since H_e and H_f are subgroups containing the idempotents e and f, respectively, $H_e \neq H_f$. This implies that $H_e \cap H_f = \emptyset$.

Theorem 5.6. *[Green's Theorem] If $a \in S$, then a lies in a subgroup iff $a \,\mathcal{H}\, a^2$.*

Proof. See later. □

Corollary 5.7. *Let $a \in S$. Then the following are equivalent:*

(i) *a lies in a subgroup,*
(ii) *$a \,\mathcal{H}\, e$, for some $e \in E(S)$,*
(iii) *H_a is a subgroup,*
(iv) *$a \,\mathcal{H}\, a^2$.*

Proof. (i) $\Rightarrow$ (ii): If $a \in G$, then $G \subseteq H_e$ where $e^2 = e$ is the identity for G. Therefore $a \in H_e$ so $a \,\mathcal{H}\, e$.

(ii) $\Rightarrow$ (iii): If $a \,\mathcal{H}\, e$, then $H_a = H_e$ and by the MST, H_e is a subgroup.

(iii) $\Rightarrow$ (i): Straightforward, for $a \in H_a$.

(iii) $\Rightarrow$ (iv) If H_a is a subgroup, then certainly H_a is closed. Hence $a, a^2 \in H_a$ therefore $a \,\mathcal{H}\, a^2$.

(iv) $\Rightarrow$ (i) This follows from Greeen's Theorem (Theorem 5.6). □

Subgroups of $\mathcal{T}_n$

We use Green's Theorem to show the following.

Lemma 5.8. *Let $\alpha \in \mathcal{T}_n$. Then α lies in a subgroup of $\mathcal{T}_n \Leftrightarrow$ the map diagram has no tails of length $\geqslant 2$.*

Proof. We have that

$$\begin{aligned} \alpha \text{ lies in a subgroup} &\Leftrightarrow \alpha \,\mathcal{H}\, \alpha^2 \\ &\Leftrightarrow \alpha \,\mathcal{L}\, \alpha^2, \alpha \,\mathcal{R}\, \alpha^2 \\ &\Leftrightarrow \operatorname{Im} \alpha = \operatorname{Im} \alpha^2, \ker \alpha = \ker \alpha^2. \end{aligned}$$

We know $\operatorname{Im} \alpha^2 \subseteq \operatorname{Im} \alpha$ (as $\mathcal{T}_n \alpha^2 \subseteq \mathcal{T}_n \alpha$). Let ρ be an equivalence on a set X. Recall

$$X/\rho = \{[x] \mid x \in X\}$$

We have seen that

$$|\underline{n}/\ker \alpha| = |\operatorname{Im} \alpha|.$$

We know that $\ker \alpha \subseteq \ker \alpha^2$ ($\alpha^2 \mathcal{T}_n \subseteq \alpha \mathcal{T}_n$), which means that the $\ker \alpha^2$-classes are just unions of $\ker \alpha$-classes:

CLAIM. For $\alpha \in \mathcal{T}_n$, $\operatorname{Im} \alpha = \operatorname{Im} \alpha^2 \Leftrightarrow \ker \alpha = \ker \alpha^2$.