Thus, idempotents are left/right/two-sided identities for their $\mathcal{R}/\mathcal{L}/\mathcal{H}$-classes.

Lemma 5.4. *Let G be a subgroup with idempotent e. Then $G \subseteq H_e$, thus, the elements of G are all $\mathcal{H}$-related.*

Proof. Let G be a subgroup with idempotent e. Then for any $a \in G$ we have $ea = a = ae$ and there exists $a^{-1} \in G$ with $aa^{-1} = e = a^{-1}a$. Then

$$\left.\begin{aligned} ea &= a \\ aa^{-1} &= e \end{aligned}\right\} \Rightarrow a \,\mathcal{R}\, e$$
$$\left.\begin{aligned} ae &= a \\ a^{-1}a &= e \end{aligned}\right\} \Rightarrow a \,\mathcal{L}\, e$$
$$\Rightarrow a \,\mathcal{H}\, e.$$

Therefore $a \,\mathcal{H}\, e$ for all $a \in G$, so $G \subseteq H_e$. □

Theorem 5.5 (Maximal Subgroup Theorem). *Let $e \in E(S)$. Then H_e is the maximal subgroup of S with identity e.*

Proof. We have shown that if G is a subgroup with identity e, then $G \subseteq H_e$.
We show now that H_e itself is a subgroup with identity e.
We know that e is an identity for H_e. Suppose $a, b \in H_e$. Then $b \,\mathcal{H}\, e$, so $b \,\mathcal{R}\, e$ hence $ab \,\mathcal{R}\, ae$ ($\mathcal{R}$ is left compatible) so

$$ab \,\mathcal{R}\, ae = a \,\mathcal{R}\, e.$$

Also, $a \,\mathcal{L}\, e \Rightarrow ab \,\mathcal{L}\, eb = b \,\mathcal{L}\, e$ hence $ab \,\mathcal{H}\, e$ so $ab \in H_e$. It remains to show that for all $a \in H_e$ there exists $b \in H_e$ with $ab = e = ba$.

Let $a \in H_e$. Then, by definition of $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, there exist $s, t \in S^1$ with

$$\underbrace{at = e}_{a\mathcal{R}e}\underbrace{= sa}_{a\mathcal{L}e}.$$

We have

$$a(ete) = (ae)te = ate = ee = e = \cdots = (ese)a.$$

Let $x = ete$, $y = ese$ so $x, y \in S$ and $ex = xe = x$, $ey = ye = y$. Also $e = ax = ya$. Now

$$x = ex = (ya)x = y(ax) = ye = y.$$

So let $b = x = y$. Then

$$\underbrace{eb = b \quad ba = e}_{b\mathcal{R}e} \qquad \underbrace{be = b \quad ab = e}_{b\mathcal{L}e}$$

so $b \,\mathcal{H}\, e$, thus $b \in H_e$. Hence H_e is indeed a subgroup. □