(5) $\mathcal{S}_X$ is a subgroup of $\mathcal{T}_X$. Notice

$$\begin{aligned} \alpha \,\mathcal{H}\, I_X &\Leftrightarrow \operatorname{Im}\alpha = \operatorname{Im} I_X \text{ and } \ker\alpha = \ker I_X, \\ &\Leftrightarrow \operatorname{Im}\alpha = X \text{ and } \ker\alpha = \iota, \\ &\Leftrightarrow \alpha \text{ is onto and } \alpha \text{ is one-one}, \\ &\Leftrightarrow \alpha \in \mathcal{S}_X. \end{aligned}$$

Therefore $\mathcal{S}_X$ is the $\mathcal{H}$-class of I_X.

Definition 5.1. In the sequel, we are going to denote by L_a the $\mathcal{L}$-class of a; by R_a the $\mathcal{R}$-class of a and by H_a the $\mathcal{H}$-class of a.

Now $L_a = L_b \Leftrightarrow a \,\mathcal{L}\, b$ and $H_a = L_a \cap R_a$. For example, in B, we have $L_{(2,3)} = \{(x, 3) \mid x \in \mathbb{N}^0\}$.

We are going to show that the maximal subgroups of semigroups are just the $\mathcal{H}$-classes of idempotents. As a consequence, we will see that whenever two subgroups are not disjoint, then they are both contained within a subgroup, as the following figure shows.

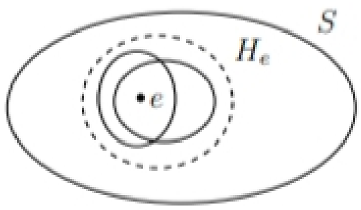

Figure 3. Existence of a Maximal Subgroup.

Lemma 5.2 (Principal Ideal for Idempotents). *Let $a \in S$, $e \in E(S)$. Then*

(i) $S^1a \subseteq S^1e \Leftrightarrow ae = a$

(ii) $aS^1 \subseteq eS^1 \Leftrightarrow ea = a$.

Proof. (We prove part (i) only because (ii) is dual). If $ae = a$, then $a \in S^1e$ so $S^1a \subseteq S^1e$ by the Principal Ideal Lemma. Conversely, if $S^1a \subseteq S^1e$ then by the Principal Ideal Lemma we have $a = te$ for some $t \in S^1$. Then

$$ae = (te)e = t(ee) = te = a.$$

□

Corollary 5.3. *Let $e \in E(S)$. Then we have*

$$\begin{aligned} a \,\mathcal{R}\, e &\Rightarrow ea = a, \\ a \,\mathcal{L}\, e &\Rightarrow ae = a, \\ a \,\mathcal{H}\, e &\Rightarrow a = ae = ea. \end{aligned}$$