Now we have $\operatorname{Im} \varepsilon = \{2,3\}$. We can see that $\ker \varepsilon$ has classes $\{1,2\}, \{3\}$. So

$$\begin{aligned} \alpha\,\mathcal{H}\,\varepsilon &\Leftrightarrow \operatorname{Im}\alpha = \operatorname{Im}\varepsilon \text{ and } \ker\alpha = \ker\epsilon \\ &\Leftrightarrow \operatorname{Im}\alpha = \{2,3\} \text{ and } \ker\alpha \text{ has classes } \{1,2\},\{3\}. \end{aligned}$$

So we have

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} \qquad \text{or} \qquad \alpha = \varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$$

	ε	α
ε	ε	α
α	α	ε

which is the table of a 2-element group. Thus the $\mathcal{H}$-class of ε is a group.

5. Subgroups of Semigroups

Let S be a semigroup and let $H \subseteq S$. Then H is a *subgroup* of S if it is a group under the restriction of the binary operation on S to H; i.e.

- $a, b \in H \Rightarrow ab \in H$
- $\exists\, e \in H$ with $ea = a = ae$ for all $a \in H$
- $\forall\, a \in H\, \exists\, b \in H$ with $ab = e = ba$

Remark.

(1) S does not have to be a monoid. Even if S *is* a monoid, e does no [illegible]. However, e must be an idempotent, i.e. $e \in E(S)$.

(2) If H is a subgroup with identity e, then e is the *only* idempotent in H.

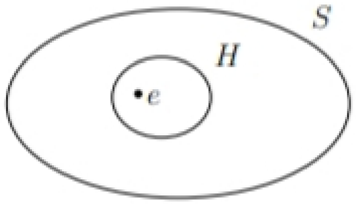

Figure 2. e is the only idempotent in H.

(3) If $e \in E(S)$, then $\{e\}$ is a trivial subgroup.

(4) With $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$ and $\epsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ we have the $\mathcal{H}$-class $\{\epsilon, \alpha\}$ is a subgroup of $\mathcal{T}_3$.