4.3. $\mathcal{L}$ and $\mathcal{R}$ in $\mathcal{T}_X$

CLAIM. $\alpha\mathcal{T}_X \subseteq \beta\mathcal{T}_X \Leftrightarrow \ker\beta \subseteq \ker\alpha$.
(Recall $\ker\alpha = \{(x,y) \in X \times X \mid x\alpha = y\alpha\}$).

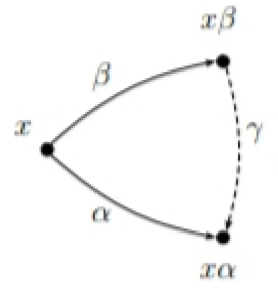
Proof. ($\Rightarrow$) Suppose $\alpha\mathcal{T}_X \subseteq \beta\mathcal{T}_X$. Then $\alpha = \beta\gamma$ for some $\gamma \in \mathcal{T}_X$. Let $(x,y) \in \ker\beta$. Then

$$x\alpha = x(\beta\gamma) = (x\beta)\gamma = (y\beta)\gamma = y(\beta\gamma) = y\alpha.$$

Hence $(x,y) \in \ker\alpha$ and so $\ker\beta \subseteq \ker\alpha$.

($\Leftarrow$) Suppose $\ker\beta \subseteq \ker\alpha$. Define $\gamma\colon X \to X$ by

$$z\gamma = \begin{cases} z & z \notin \operatorname{Im}\beta \\ x\alpha & z = x\beta \end{cases}$$



If $z = x\beta = y\beta$, then $(x,y) \in \ker\beta \subseteq \ker\alpha$ so $x\alpha = y\alpha$. Hence γ is well-defined. So $\gamma \in \mathcal{T}_X$ and $\beta\gamma = \alpha$. Therefore $\alpha \in \beta\mathcal{T}_X$ so that by the Principal Ideal Lemma, $\alpha\mathcal{T}_X \subseteq \beta\mathcal{T}_X$. □

Corollary 4.19 ($\mathcal{R}$–$\mathcal{T}_X$-Lemma). $\alpha\,\mathcal{R}\,\beta \Leftrightarrow \ker\alpha = \ker\beta$.

Proof. We have

$$\begin{aligned} \alpha\,\mathcal{R}\,\beta &\Leftrightarrow \alpha\mathcal{T}_X = \beta\mathcal{T}_X \\ &\Leftrightarrow \alpha\mathcal{T}_X \subseteq \beta\mathcal{T}_X \text{ and } \beta\mathcal{T}_X \subseteq \alpha\mathcal{T}_X \\ &\Leftrightarrow \ker\beta \subseteq \ker\alpha \text{ and } \ker\alpha \subseteq \ker\beta \\ &\Leftrightarrow \ker\alpha = \ker\beta. \end{aligned}$$

□

FACT: $\mathcal{T}_X\alpha \subseteq \mathcal{T}_X\beta \Leftrightarrow \operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$ (See Exercises).

Corollary 4.20 ($\mathcal{L} - \mathcal{T}_X$-Lemma). $\alpha\,\mathcal{L}\,\beta \Leftrightarrow \operatorname{Im}\alpha = \operatorname{Im}\beta$.

Consequently $\alpha\,\mathcal{H}\,\beta \Leftrightarrow \ker\alpha = \ker\beta$ and $\operatorname{Im}\alpha = \operatorname{Im}\beta$.

EXAMPLE 4.21. Let us define

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \in E(\mathcal{T}_3)$$