(2) If $Sa = S^1a$ and $Sb = S^1b$, then $a \mathcal{L} b \Leftrightarrow \exists\, s, t \in S$ with $a = sb$, $b = ta$.

Dually, the relation $\mathcal{R}$ is defined on S by

$$a \mathcal{R} b \Leftrightarrow aS^1 = bS^1$$

and

$$\begin{aligned} a \mathcal{R} b &\Leftrightarrow \exists\, s, t \in S^1 \text{ with } a = bs \text{ and } b = at, \\ &\Leftrightarrow a = b \text{ or } \exists\, s, t \in S \text{ with } a = bs \text{ and } b = at. \end{aligned}$$

We can adjust this if $aS^1 = aS$ as before. Now $\mathcal{R}$ is an *equivalence*; it is *left compatible* and hence a *left congruence*.

Definition 4.15. We define the relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and note that $\mathcal{H}$ is an equivalence.

The relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ are in fact three of the so-called *Greens' relations*.

Example 4.16. (1) If S is commutative, $\mathcal{L} = \mathcal{R} = \mathcal{H}$.
(2) In a group G,

$$G^1a = G = G^1b \quad \text{and} \quad aG^1 = G = bG^1 \quad \text{for all } a, b \in G.$$

So $a \mathcal{L} b$ and $a \mathcal{R} b$ for all $a, b \in G$. Therefore $\mathcal{L} = \mathcal{R} = \omega = G \times G$ and hence we have $\mathcal{H} = \omega$.

Example 4.17. In $\mathbb{N}$ under $+$ we have

$$a + \mathbb{N}^1 = \{a, a+1, \dots\}$$

and so $a + \mathbb{N}^1 = b + \mathbb{N}^1 \Leftrightarrow a = b$. Hence $\mathcal{L} = \mathcal{R} = \mathcal{H} = \iota$.

Example 4.18. In B we know

$$(m, n)B^1 = \{(x, y) \mid x \geqslant m, y \in \mathbb{N}^0\}$$

and so we have

$$(m, n)B^1 = (p, q)B^1 \Leftrightarrow m = p.$$

Hence $(m, n) \mathcal{R} (p, q) \Leftrightarrow m = p$. Dually,

$$(m, n) \mathcal{L} (p, q) \Leftrightarrow n = q.$$

Thus $(m, n) \mathcal{H} (p, q) \Leftrightarrow (m, n) = (p, q)$, which gives us $\mathcal{H} = \iota$.