iii) *$a = tb$ for some $t \in S^1$,*
iv) *$a = b$ or $a = tb$ for some $t \in S$.*

NOTE. If $S^1a = Sa$ and $S^1b = Sb$, then the Lemma can be adjusted accordingly.

Proof. It is clear that (ii), (iii) and (iv) are equivalent.

(i) $\Rightarrow$ (ii): If $S^1a \subseteq S^1b$ then $a = 1a \in S^1a \subseteq S^1b \Rightarrow a \in S^1b$.

(ii) $\Rightarrow$ (i): If $a \in S^1b$, then as S^1a is the smallest left ideal containing a, and as S^1b is a left ideal we have $S^1a \subseteq S^1b$. □

Lemma 4.12 (Principal Right Ideal Lemma). *The following statements are equivalent:*
i) *$aS^1 \subseteq bS^1$,*
ii) *$a \in bS^1$,*
iii) *$a = bt$ for some $t \in S^1$,*
iv) *$a = b$ or $a = bt$ for some $t \in S$.*

NOTE. If $aS = aS^1$ and $bS = bS^1$ then $aS \subseteq bS \Leftrightarrow a \in bS \Leftrightarrow a = bt$ for some $t \in S$.

The following relation is crucial in semigroup theory.

DEFINITION 4.13. The relation $\mathcal{L}$ on a semigroup S is defined by the rule

$$a \,\mathcal{L}\, b \Leftrightarrow S^1a = S^1b$$

for any $a, b \in S$.

NOTE.
(1) $\mathcal{L}$ is an equivalence.
(2) If $a \,\mathcal{L}\, b$ and $c \in S$ then $S^1a = S^1b$, so $S^1ac = S^1bc$ and hence $ac \,\mathcal{L}\, bc$, i.e. $\mathcal{L}$ is right compatible. We call a right (left) compatible equivalence relation a *right (left) congruence.* Thus $\mathcal{L}$ is a right congruence.

Corollary 4.14. *We have that*

$$\boxed{a \,\mathcal{L}\, b \Leftrightarrow \exists\, s, t \in S^1 \text{ with } a = sb \text{ and } b = ta.}$$

Proof.

$$\begin{aligned} a \,\mathcal{L}\, b &\Leftrightarrow S^1a = S^1b \\ &\Leftrightarrow S^1a \subseteq S^1b \text{ and } S^1b \subseteq S^1a \\ &\Leftrightarrow \exists\, s, t \in S^1 \text{ with } a = sb, b = ta \end{aligned}$$

by the Principal Left Ideal Lemma. □

We note that this statement about $\mathcal{L}$ **can be used as a definition of $\mathcal{L}$.**

REMARK.
(1) $a \,\mathcal{L}\, b \Leftrightarrow a = b$ or there exist $s, t \in S$ with $a = sb$, $b = ta$.