$$S^1aS^1 = SaS \cup aS \cup Sa \cup \{a\}.$$

Claim. aS^1 (S^1a, S^1aS^1) is the "smallest" right (left, two-sided ideal) containing a.

Proof. (for aS^1).
We have $a = a1 \in aS^1$ and $(aS^1)S = a(S^1S) \subseteq aS^1$. So, aS^1 is a right ideal containing a. If $a \in I$ and I is a right ideal, then $aS^1 \subseteq IS^1 = I \cup IS \subseteq I$. □

Definition 4.7. We call aS^1 (S^1a, S^1aS^1) the *principal right (left, two-sided) ideal generated by a.*

If S is commutative then $aS^1 = S^1a = S^1aS^1$.

Example 4.8. In a group G we have

$$aG^1 = G = G^1a = G^1aG^1$$

for all $a \in G$.

Example 4.9. In $\mathbb{N}$ under addition we have

$$n + \text{``}\mathbb{N}^1\text{''} = I_n = \{n, n+1, n+2, \dots\}$$

Example 4.10. B is simple, so

$$B(m,n)B = B^1(m,n)B^1 = B$$

for all $(m,n) \in B$. However:

Claim. $(m,n)B = (m,n)B^1 = \big\{(x,y) \mid x \geqslant m, y \in \mathbb{N}^0\big\}$

Proof. We have

$$\begin{aligned}(m,n)B &= \big\{(m,n)(u,v) \mid (u,v) \in B\big\} \\ &\subseteq \big\{(x,y) \mid x \geqslant m, y \in \mathbb{N}^0\big\}.\end{aligned}$$

Let $x \geqslant m$ then

$$\begin{aligned}(m,n)\big(n+(x-m),y\big) &= \big(m-n+n+(x-m),y\big), \\ &= (x,y).\end{aligned}$$

Therefore $(x,y) \in (m,n)B \Rightarrow \big\{(x,y) \mid x \geqslant m, y \in \mathbb{N}^0\big\} \subseteq (m,n)B$. Hence we have proved our claim. □

Dually we have $B(m,n) = \big\{(x,y) \mid x \in \mathbb{N}^0, y \geqslant n\big\}$.

Lemma 4.11 (Principal Left Ideal Lemma). *The following statements are equivalent;*

i) $S^1a \subseteq S^1b$,
ii) $a \in S^1b$,