EXAMPLE 4.4. Let G be a group and I a left ideal. Let $g \in G, a \in I$ then we have

$$g = (ga^{-1})a \in I$$

and so $G = I$. Therefore G has no proper left/right ideals. Hence G is simple.

Exercise: G^0 is 0-simple

EXAMPLE 4.5. We have $(\mathbb{N}, +)$ is a semigroup. Let $n \in \mathbb{N}$. Now define $I_n \subseteq (\mathbb{N}, +)$ to be

$$I_n = \{n, n+1, n+2, \dots\},$$

which is an ideal. Hence $\mathbb{N}$ is not simple.

NOTE. $\{2, 4, 6, \dots\}$ is a subsemigroup but *not* an ideal.

EXAMPLE 4.6. The bicyclic semigroup B is simple.

Proof. Let $I \subseteq B$ be an ideal, say $(m, n) \in I$. Then $(0, n) = (0, m)(m, n) \in I$. Thus $(0, 0) = (0, n)(n, 0) \in I$. Let $(a, b) \in B$. Then

$$(a, b) = (a, b)(0, 0) \in I$$

and hence $B = I \Rightarrow B$ is simple. □

4.2. Principal Ideals

We make note of how the S^1 notation can be used. For example

$$\begin{aligned} S^1A &= \{sa \mid s \in S^1, a \in A\}, \\ &= \{sa \mid s \in S \cup \{1\}, a \in A\}, \\ &= \{sa \mid s \in S, a \in A\} \cup \{1a \mid a \in A\}, \\ &= SA \cup A. \end{aligned}$$

In particular, if $A = \{a\}$ then $S^1a = Sa \cup \{a\}$. So,

$$\begin{aligned} S^1a = Sa &\Leftrightarrow a \in Sa, \\ &\Leftrightarrow a = ta \end{aligned}$$

for some $t \in S$. We have $S^1a = Sa$ for $a \in S$ if:

- S is a *monoid* (then $a = 1a$).
- $a \in E(S)$ (then $a = aa$).
- a is *regular*, i.e. there exists $x \in S$ with $a = axa$ (then $a = (ax)a$).

But in $(\mathbb{N}, +)$ we have $1 \notin 1 + \mathbb{N}$. Dually,

$$aS^1 = aS \cup \{a\}$$

and similarly