(3) $AC = BC \not\Rightarrow A = B$ and $CA = CB \not\Rightarrow A = B$, i.e. the power semigroup is not cancellative - think of a right zero semigroup, there $AC = BC = C$ for all $A, B, C \subseteq S$.
(4) A is isomorphic to the subsemigroup $\{\{a\} \mid a \in A\}$ of $\mathcal{P}(A)$.
(5) $S^1S = S = SS^1$.

DEFINITION 4.1. Let $\emptyset \neq I \subseteq S$ then I is
(1) a *left ideal* if $SI \subseteq I$ (i.e. $a \in I, s \in S \Rightarrow sa \in I$);
(2) a *right ideal* if $IS \subseteq I$;
(3) an *(two-sided) ideal* if $IS \cup SI \subseteq I$, that is, I is both a left and a right ideal.

Note that if S is commutative, (1),(2) and (3) above coincide.
If $\emptyset \neq I \subseteq S$ then we have:
I is a left ideal $\Leftrightarrow S^1I \subseteq I$;
I is a right ideal $\Leftrightarrow IS^1 \subseteq I$;
I is an ideal $\Leftrightarrow S^1IS^1 \subseteq I$.

Note that any (left/right) ideal is a subsemigroup.

EXAMPLE 4.2. (1) Let $i \in I$ then $\{i\} \times J$ is a right ideal in a rectangular band $I \times J$.
(2) Let $m \in \mathbb{N}^0$ be fixed. Then $I_m = \{(x, y) \mid x \geqslant m, y \in \mathbb{N}^0\}$ is a right ideal in the bicyclic semigroup B.
Indeed, let $(x, y) \in I_m$ and let $(a, b) \in B$. Then

$$(x, y)(a, b) = (x - y + t, b - a + t),$$

where $t = \max\{y, a\}$. Now, we know that $x \geq m$ and that $t \geq y$, so $t - y \geq 0$. Adding up these two inequalities, we get that $x - y + t \geq m$, thus the product is indeed in I_m.
(3) If $Y \subseteq X$ then we have $\{\alpha \in \mathcal{T}_X \mid \operatorname{Im} \alpha \subseteq Y\}$ is a left ideal of $\mathcal{T}_X$.
(4) For any $n \in \mathbb{N}$ we define

$$S^n = \{a_1a_2 \dots a_n \mid a_i \in S\}.$$

This is an ideal of S. If S is a monoid then $S^n = S$ for all n, since for any $s \in S$ we can write

$$s = s\underbrace{11\dots1}_{n-1} \in S^n.$$

(5) If S has a zero 0, then $\{0\}$ (usually written 0), is an ideal.

DEFINITION 4.3. Let S be a semigroup.
(1) We say that S is *simple* if S is the only ideal.
(2) If S has a zero 0, then S is *0-simple* if S and $\{0\}$ are the only ideals and $S^2 \neq 0$.

Note that S^2 is always an ideal, so the condition $S^2 \neq 0$ is only required to exclude the 2-element null semigroup. A null semigroup is a semigroup with zero such that every product equals 0 - notice that every subset containing 0 is an ideal.