$$\begin{aligned}[a] = [b] &\Leftrightarrow a \ker\theta\, b \\ &\Leftrightarrow a\theta = b\theta \\ &\Leftrightarrow [a]\bar{\theta} = [b]\bar{\theta}.\end{aligned}$$

Hence $\bar{\theta}$ is well-defined and one-one. For any $x \in \operatorname{Im}\theta$ we have $x = a\theta = [a]\bar{\theta}$ and so $\bar{\theta}$ is onto. Finally,

$$([a][b])\bar{\theta} = [ab]\bar{\theta} = (ab)\theta = a\theta b\theta = [a]\bar{\theta}[b]\bar{\theta}.$$

Therefore $\bar{\theta}$ is an isomorphism and $S/\ker\theta \cong \operatorname{Im}\theta$. □

Note that the analogue of Theorem 3.8 holds for monoid to give us the **The Fundamental Theorem of Morphisms for Monoids**.

EXAMPLE 3.9. $\theta : B \to (\mathbb{Z}, +)$ given by $(a, b)\theta = a - b$ is a monoid morphism. Check that θ is onto, so by FTH we have

$$B/\ker\theta \cong \mathbb{Z}.$$

Moreover, $\ker\theta$ is the congruence given by

$$(a, b) \ker\theta\, (c, d) \Leftrightarrow a - b = c - d.$$

4. Ideals

Ideals play an important role in Semigroup Theory, but rather different to that they hold in Ring Theory. The reason is that in case of rings, ALL homomorphisms are determined by ideals, but in case of semigroups, only some *are.*

4.1. Notation

If $A, B \subseteq S$ then we write

$$AB = \{ab \mid a \in A, b \in B\},$$
$$A^2 = AA = \{ab \mid a, b \in A\}.$$

NOTE. A is a subsemigroup if and only if $A \neq \emptyset$ and $A^2 \subseteq A$.

We write aB for $\{a\}B = \{ab \mid b \in B\}$.
For example

$$AaB = \{xay \mid x \in A, y \in B\}.$$

Facts:

(1) $A(BC) = (AB)C$ therefore $\mathcal{P}(S) = \{S \mid A \subseteq S\}$, equipped by the above-defined operation, is a semigroup – the *power semigroup* of S.
(2) $A \subseteq B \Rightarrow AC \subseteq BC$ and $CA \subseteq CB$ for all $A, B, C \in \mathcal{P}(S)$.