as ρ is a congruence we have $ab\ \rho\ a'b'$ and hence $[ab] = [a'b']$. Hence our operation is well-defined. Let $[a], [b], [c] \in S/\rho$ then we have

$$\begin{aligned}
[a]\big([b][c]\big) &= [a][bc], \\
&= \big[a(bc)\big], \\
&= \big[(ab)c\big], \\
&= [ab][c], \\
&= \big([a][b]\big)[c].
\end{aligned}$$

If S is a monoid, then so is S/ρ because we have

$$[1][a] = [1a] = [a] = [a1] = [a][1]$$

for any $a \in S$. Hence we conclude that S/ρ is a semigroup and if S is a monoid, then so is S/ρ.

Definition 3.7. We call S/ρ the *factor semigroup* (or monoid) of S by ρ.

Now, define $\nu_\rho : S \to S/\rho$ by

$$s\nu_\rho = [s].$$

Then we have

$$\begin{aligned}
s\nu_\rho t\nu_\rho &= [s][t] && \text{definition of } \nu_\rho, \\
&= [st] && \text{definition of multiplication in } S/\rho, \\
&= (st)\nu_\rho && \text{definition of } \nu_\rho.
\end{aligned}$$

Hence ν_ρ is a semigroup morphism. Moreover if S is a monoid then $1\nu_\rho = [1]$, so that ν_ρ is a monoid morphism. We now want to examine the kernel of ν_ρ:

$$\begin{aligned}
s \ker \nu_\rho\ t &\Leftrightarrow s\nu_\rho = t\nu_\rho && \text{definition of } \ker \nu_\rho, \\
&\Leftrightarrow [s] = [t] && \text{definition of } \nu_\rho, \\
&\Leftrightarrow s\ \rho\ t && \text{definition of } \rho.
\end{aligned}$$

Therefore $\rho = \ker \nu_\rho$ and so every congruence is the kernel of a morphism.

Theorem 3.8. *[The Fundamental Theorem of Morphisms for Semigroups] Let $\theta\colon S \to T$ be a semigroup morphism. Then $\ker\theta$ is a congruence on S, $\operatorname{Im}\theta$ is a subsemigroup of T and $S/\ker\theta \cong \operatorname{Im}\theta$.*

Proof. Define $\bar{\theta}\colon S/\ker\theta \to \operatorname{Im}\theta$ by $[a]\bar{\theta} = a\theta$. We have