Note. Some remarks on the notion *well-defined*: usually we define a map on a set by simply stating what the image of the individual elements should be, e.g:

$\alpha\colon \mathbb{N} \to \mathbb{Z},\ n\alpha =$ the number of 9's less the number of 2's in the decimal form of n.

But very often in mathematics, the set on which we would like to define the map is a set of classes of an equivalence relation (that is, the *factor set of the relation*). In such cases, we usually define the map by using the elements of the equivalence classes (for usually we can use some operations on them). For example let

$$\rho = \{(n, m) | n \equiv m \pmod 4\} \subseteq \mathbb{N} \times \mathbb{N}.$$

Then ρ is an equivalence relation having the following 4 classes:

$$A = \{1, 5, 9, 13, \ldots\}, B = \{2, 6, 10, 14, \ldots\},$$

$$C = \{3, 7, 11, 15, \ldots\}, D = \{4, 8, 12, 16, \ldots\}.$$

Thus, the factor set of ρ is $X = \{A, B, C, D\}$. We try do define a map from X to $\mathbb{N}$ by

$$\alpha\colon X \to \mathbb{N}, [n]_\rho\alpha = 2^n.$$

What is the image of A under α? We choose an element n of A (that is, we *represent* A as $[n]_\rho$): $1 \in A$, thus $A = [1]_\rho$. So $A\alpha = [1]_\rho\alpha = 2$. However, $5 \in A$, too! So we have $A\alpha = [5]_\rho\alpha = 2^5 = 32$. Thus, $A\alpha$ has more than one values. We refer to this situation as 'α being not well-defined'.

Keep in mind that whenever we try to define something (a map, or an operation) on a factor set of an equivalence relation by referring to ELEMENTS of the equivalence classes, it MUST be checked, that the choice of the elements of the equivalence classes does not influence the result.

For example in the above-mentioned example let

$$\beta\colon X \to \mathbb{N}^0, [n]_\rho\beta = \overline{n},$$

where $\overline{n}$ denotes the remainder of n on division by 4 (that is, $0, 1, 2$ or 3). In this case β is well-defined, because all elements in the same class have the same remainder, for example

$$A\beta = [1]_\rho\beta = 1 = [5]_\rho\beta = [9]_\rho\beta = \ldots$$

The following construction and lemmas might be familiar...

Let ρ be a congruence on S. Then we define

$$S/\rho = \big\{[a] \mid a \in S\big\}.$$

Define a binary operation on S/ρ by

$$[a][b] = [ab].$$

We need to make sure that this is a well-defined operation, that is, that the product $[a][b]$ does not depend on the choice of a and b. If $[a] = [a']$ and $[b] = [b']$ then $a\ \rho\ a'$ and $b\ \rho\ b'$;