Recall that

$$[a] = \{b \in A \mid a \;\rho\; b\}.$$

If $\rho$ is an equivalence relation then $[a]$ is the equivalence-class, or the $\rho$-class, of $a$.

We denote by $\omega$ the UNIVERSAL relation on $A$: $\omega = A \times A$. So $x \;\omega\; y$ for all $x, y \in A$, and $[x] = A$ for all $x \in A$.

We denote by $\iota$ be the EQUALITY relation on $A$:

$$\iota = \big\{(a, a) \mid a \in A\big\}.$$

Thus $x \;\iota\; y \Leftrightarrow x = y$ and so $[x] = \{x\}$ for all $x \in A$.

## 3.2. Algebra of Relations

If $\rho, \lambda$ are relations on $A$, then so is $\rho \cap \lambda$. For all $a, b \in A$ we have

$$\begin{aligned} a\;(\rho \cap \lambda)\; b &\Leftrightarrow (a, b) \in \rho \cap \lambda \\ &\Leftrightarrow (a, b) \in \rho \text{ and } (a, b) \in \lambda \\ &\Leftrightarrow a \;\rho\; b \text{ and } a \;\lambda\; b. \end{aligned}$$

We note that $\rho \subseteq \lambda$ means $a \;\rho\; b \Rightarrow a \;\lambda\; b$.

Note that $\iota \subseteq \rho \Leftrightarrow \rho$ is reflexive and so $\iota \subseteq \rho$ for any equivalence relation $\rho$.

We see that $\iota$ is the smallest equivalence relation on $A$ and $\omega$ is the largest equivalence relation on $A$.

**Lemma 3.2.** *If $\rho, \lambda$ are equivalence relations on $A$ then so is $\rho \cap \lambda$.*

*Proof.* We have $\iota \subseteq \rho$ and $\iota \subseteq \lambda$, then $\iota \subseteq \rho \subseteq \lambda$, so $\rho \cap \lambda$ is reflexive. Suppose $(a, b) \in \rho \cap \lambda$. Then $(a, b) \in \rho$ and $(a, b) \in \lambda$. So as $\rho$, $\lambda$ are symmetric, we have $(b, a) \in \rho$ and $(b, a) \in \lambda$ and hence $(b, a) \in \rho \cap \lambda$. Therefore $\rho \cap \lambda$ is symmetric. By a similar argument we have $\rho \cap \lambda$ is transitive. Therefore $\rho \cap \lambda$ is an equivalence relation. $\square$

Denoting by $[a]_\rho$ the $\rho$-class of $a$ and $[a]_\lambda$ the $\lambda$-class of $a$ we have that,

$$\begin{aligned} [a]_{\rho \cap \lambda} &= \{b \in A \mid b \;\rho \cap \lambda\; a\}, \\ &= \{b \in A \mid b \;\rho\; a \text{ and } b \;\lambda\; a\}, \\ &= \{b \in A \mid b \;\rho\; a\} \cap \{b \in A \mid b \;\lambda\; a\}, \\ &= [a]_\rho \cap [a]_\lambda. \end{aligned}$$

We note that $\rho \cup \lambda$ need not be an equivalence relation. On $\mathbb{Z}$ we have