Now there is only one idempotent such that $|\operatorname{Im}\varepsilon| = 3$, that is the identity map

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

3. Relations

Please see the handout 'Functions and Relations'.

In group theory, homomorphic images of groups are determined by nor[illegible]s. The situation is more complicated in semigroup theory, namely the homomorphic images of semigroups are determined by special equivalence relations. Furthermore, elements of semigroups can be quite often 'ordered'. For example there is a natural notion of a map being 'bigger' than another one: namely if its image has a bigger cardinality. These examples show that relations play a central role in semigroup theory.

Definition 3.1. A (binary) *relation* ρ on A is a subset of $A \times A$.

Convention: we may write "$a \, \rho \, b$" for "$(a, b) \in \rho$".

3.1. Some special relations

Properties of the relation $\leqslant$ on $\mathbb{R}$:

$$\begin{array}{cl} a \leqslant a & \text{for all } a \in \mathbb{R}, \\ a \leqslant b \text{ and } b \leqslant c \Rightarrow a \leqslant c & \text{for all } a, b, c \in \mathbb{R}, \\ a \leqslant b \text{ and } b \leqslant a \Rightarrow a = b & \text{for all } a, b \in \mathbb{R}, \\ a \leqslant b \text{ or } b \leqslant a & \text{for all } a, b \in \mathbb{R}. \end{array}$$

Thus, the relation $\leqslant$ is a total order on $\mathbb{R}$ (sometimes we say that $\mathbb{R}$ is linearly ordered by $\leqslant$).

Recall that if X is any set, we denote by $\mathcal{P}(X)$ the set of all subsets of X (and call it the *power set of* X. Properties of the relation $\subseteq$ on a power set $\mathcal{P}(X)$ of an arbitrary set X:

$$\begin{array}{cl} A \subseteq A & \text{for all } A \in \mathcal{P}(X) \\ A \subseteq B \text{ and } B \subseteq C \Rightarrow A \subseteq C & \text{for all } A, B, C \in \mathcal{P}(X) \\ A \subseteq B \text{ and } B \subseteq A \Rightarrow A = B & \text{for all } A, B \in \mathcal{P}(X) \end{array}$$

Notice that if $|X| > 2$ and $x, y \in X$ with $x \neq y$ then $\{x\} \not\subseteq \{y\}$ and $\{y\} \not\subseteq \{x\}$, thus $\subseteq$ is a partial order but not a total order on $\mathcal{P}(X)$.