EXAMPLE 2.21. In $\mathcal{T}_3$ we have $\operatorname{Im} c_1 = \{1\}$, $\operatorname{Im} I_3 = \{1,2,3\}$ and

$$\operatorname{Im}\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 3\end{pmatrix} = \{2,3\}.$$

The following lemma gives a rather useful characterization of the idempotents of a transformation monoid.

Lemma 2.22 (The $E(\mathcal{T}_X)$ Lemma). *An element $\varepsilon \in \mathcal{T}_X$ is idempotent $\Leftrightarrow \varepsilon|_{\operatorname{Im}\varepsilon} = I_{\operatorname{Im}\varepsilon}$.*

Proof. $\varepsilon|_{\operatorname{Im}\varepsilon} = I_{\operatorname{Im}\varepsilon}$ means that for all $y \in \operatorname{Im}\varepsilon$ we have $y\varepsilon = y$.
Note that $\operatorname{Im}\varepsilon = \{x\varepsilon : x \in X\}$.
Then

$$\begin{aligned}
\varepsilon \in E(\mathcal{T}_X) &\Leftrightarrow \varepsilon^2 = \varepsilon, & \\
&\Leftrightarrow x\varepsilon^2 = x\varepsilon & \text{for all } x \in X,\\
&\Leftrightarrow (x\varepsilon)\varepsilon = x\varepsilon & \text{for all } x \in X,\\
&\Leftrightarrow y\varepsilon = y & \text{for all } y \in \operatorname{Im}\varepsilon,\\
&\Leftrightarrow \varepsilon|_{\operatorname{Im}\varepsilon} = I_{\operatorname{Im}\varepsilon}. &
\end{aligned}$$

□

EXAMPLE 2.23. Let

$$\alpha = \begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 3\end{pmatrix} \in \mathcal{T}_3,$$

this has image $\operatorname{Im}\alpha = \{2,3\}$. Now we can see that $2\alpha = 2$ and $3\alpha = 3$. Hence $\alpha \in E(\mathcal{T}_3)$.

EXAMPLE 2.24. We can similarly create another idempotent in $\mathcal{T}_7$, first we determine its image: let it be the subset $\{1,2,5,7\}$. Our map must fix these elements, but can map the other elements to any of these:

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 1 & 2 & & & 5 & & 7\end{pmatrix} \to \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 1 & 2 & 5 & 7 & 5 & 2 & 7\end{pmatrix} \in E(\mathcal{T}_7).$$

Using Lemma 2.22 we can now list all the idempotents in $\mathcal{T}_3$. We start with the constant maps, i.e. $\varepsilon \in E(\mathcal{T}_3)$ such that $|\operatorname{Im}\varepsilon| = 1$. These are

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 2\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\ 3 & 3 & 3\end{pmatrix}.$$

Now consider all elements $\varepsilon \in E(\mathcal{T}_3)$ such that $|\operatorname{Im}\varepsilon| = 2$. These are

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 3\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 3\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 3\end{pmatrix},$$

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 3\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 2\end{pmatrix}.$$