Proof. Let n, r be the index and period of a. Choose $s \in \mathbb{N}^0$ with $s \equiv -n \pmod{r}$. Then $s + n \equiv 0 \pmod{r}$ and so $s + n = kr$ for $k \in \mathbb{N}$. Then

$$(a^{n+s})^2 = a^{n+n+s+s} = a^{n+kr+s} = a^{n+s}$$

and so $a^{n+s} \in E(S)$. □

In fact, $\{a^n, a^{n+1}, \ldots, a^{n+r-1}\}$ is a cyclic group with identity a^{n+s}.

Corollary 2.17. *Any finite semigroup contains an idempotent.*

2.2. Idempotents in $\mathcal{T}_X$

We know $c_x c_y = c_y$ for all $x, y \in X$ and hence $c_x c_x = c_x$ for all $x \in X$. Therefore $c_x \in E(\mathcal{T}_X)$ for all $x \in X$. But if $|X| > 1$ then there are other idempotents in $\mathcal{T}_X$ as well.

Example 2.18. Let us define an element

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \in E(\mathcal{T}_X).$$

Then

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix},$$

thus α is an idempotent.

Definition 2.19. Let $\alpha \colon X \to Y$ be a map and let $Z \subseteq X$. Then the *restriction of α to the set Z* is the map

$$\alpha|_Z \colon Z \to Y, z \mapsto z\alpha \text{ for every } z \in Z.$$

NOTE: Sometimes we treat the restriction $\alpha|_Z$ as a map with domain Z and codomain $Z\alpha$.

Example 2.20. Let us define a map with domain $\{a, b, c, d\}$ and codomain $\{1, 2, 3\}$:

$$\alpha = \begin{pmatrix} a & b & c & d \\ 1 & 3 & 1 & 2 \end{pmatrix}.$$

Then $\alpha|_{\{a,d\}}$ is the following map:

$$\alpha|_{\{a,d\}} = \begin{pmatrix} a & d \\ 1 & 2 \end{pmatrix}.$$

We can see that α is *not* one-to-one but $\alpha|_{\{a,d\}}$ is.

Let $\alpha \in \mathcal{T}_X$ (i.e. $\alpha \colon X \to X$). Recall that

$$\operatorname{Im} \alpha = \{x\alpha : x \in X\} \subseteq X = X\alpha.$$