Definition 2.10. If $E(S) = S$, then S is a *band*.

Definition 2.11. If $E(S) = S$ and S is commutative, then S is a *semilattice*.

Lemma 2.12. *Let $E(S) \neq \emptyset$ and suppose $ef = fe$ for all $e, f \in E(S)$. Then $E(S)$ is a subsemigroup of S.*

Proof. Let $e, f \in E(S)$. Then

$$(ef)^2 = (ef)(ef) = e(fe)f = e(ef)f = (ee)(ff) = ef$$

and hence $ef \in E(S)$. □

From Lemma 2.12 if $E(S) \neq \emptyset$ and idempotents in S commute then $E(S)$ is a semilattice.

Example 2.13. (1) $E(B) = \{(a,a) \mid a \in \mathbb{N}^0\}$ is a semilattice.
(2) A rectangular band $I \times J$ is *not* a semilattice (unless $|I| = |J| = 1$) since $(i,j)(k,\ell) = (k,\ell)(i,j) \Leftrightarrow i = k$ and $j = \ell$.

Definition 2.14. Let $a \in S$. Then we define $\langle a \rangle = \{a^n \mid n \in \mathbb{N}\}$, which is a commutative subsemigroup of S. We call $\langle a \rangle$ the *monogenic* subsemigroup of S *generated* by a.

Proposition 2.15. *Let $a \in S$. Then either*
(i) $|\langle a \rangle| = \infty$ and $\langle a \rangle \cong (\mathbb{N}, +)$ or
(ii) $\langle a \rangle$ is finite. In this case $\exists n, r \in \mathbb{N}$ such that

$$\langle a \rangle = \{a, a^2, \ldots, a^{n+r-1}\}, |\langle a \rangle| = n + r - 1$$

$\{a^n, a^{n+1}, \ldots, a^{n+r-1}\}$ is a subsemigroup of $\langle a \rangle$ and for all $s, t \in \mathbb{N}^0$,

$$a^{n+s} = a^{n+t} \Leftrightarrow s \equiv t \ (\text{mod } r).$$

Proof. If $a^i \neq a^j$ for all $i, j \in \mathbb{N}$ with $i \neq j$ then $\theta : \langle a \rangle \to \mathbb{N}$ defined by $a^i\theta = i$ is an isomorphism. This is case (i).

Suppose that in the list of elements $a, a^2, a^3, \ldots$ there is a repetition, i.e. $a^i = a^j$ for some $i < j$. Let k be *least* such that $a^k = a^n$ for some $n < k$. Then $k = n + r$ for some $r \in \mathbb{N}$ – where n is the *index* of a, r is the *period* of a. Then the elements $a, a^2, a^3, \ldots, a^{n+r-1}$ are all distinct and $a^n = a^{n+r}$.
DO NOT CANCEL
Let $s, t \in \mathbb{N}^0$ with

$$s = s' + ur, t = t' + vr$$

with

$$0 \leq s', t' \leq r - 1, \ u, v \in \mathbb{N}^0.$$