Theorem 2.6 (The "Cayley Theorem" – for Semigroups). *Let S be a semigroup. Then S is embedded in $\mathcal{T}_{S^1}$.*

Proof. Let S be a semigroup and set $X = S^1$. We need a 1:1 morphism $S \to \mathcal{T}_X$.
For $s \in S$, we define $\rho_s \in \mathcal{T}_X$ by $x\rho_s = xs$.
Now define $\alpha : S \to \mathcal{T}_X$ by $s\alpha = \rho_s$.
We show α is 1:1: If $s\alpha = t\alpha$ then $\rho_s = \rho_t$ and so $x\rho_s = x\rho_t$ for all $x \in S^1$; in particular for $1\rho_s = 1\rho_t$ and so $1s = 1t$ hence $s = t$ and α is 1:1.

We show α is a morphism: Let $u, v \in S$. For any $x \in X$ we have

$$x(\rho_u\rho_v) = (x\rho_u)\rho_v = (xu)\rho_v = (xu)v = x(uv) = x\rho_{uv}.$$

Hence $\rho_u\rho_v = \rho_{uv}$ and so $u\alpha v\alpha = \rho_u\rho_v = \rho_{uv} = (uv)\alpha$. Therefore α is a morphism.
Hence $\alpha : S \to \mathcal{T}_X$ is an embedding. □

Theorem 2.7 (The "Cayley Theorem" - for Monoids). *Let S be a monoid. Then there exists an embedding $S \hookrightarrow \mathcal{T}_S$.*

Proof. $S^1 = S$ so $\mathcal{T}_S = \mathcal{T}_{S^1}$. We know α is a semigroup embedding. We need only check $1\alpha = I_X$.
Now $1\alpha = \rho_1$ and for all $x \in X = S$ we have

$$x\rho_1 = x1 = x = xI_X$$

and so $1\alpha = \rho_1 = I_X$. □

Theorem 2.8 (The Cayley Theorem - for Groups). *Let S be a group. Then there exists an embedding $S \hookrightarrow \mathcal{S}_S$.*

Proof. Exercise. □

2.1. Idempotents

S will always denote a semigroup.

Definition 2.9. $e \in S$ is an *idempotent* if $e^2 = e$. We put

$$E(S) = \{e \in S \mid e^2 = e\}.$$

Now, $E(S)$ may be empty, e.g. $E(S) = \emptyset$ ($\mathbb{N}$ under $+$).
$E(S)$ may also be S. If $S = I \times J$ is a rectangular band then for any $(i, j) \in S$ we have $(i, j)^2 = (i, j)(i, j) = (i, j)$ and so $E(S) = S$.

For the bicyclic semigroup B we have from Ex. 1

$$E(B) = \{(a, a) \mid a \in \mathbb{N}^0\}.$$

If S is a monoid then $1 \in E(S)$.
If S is a cancellative monoid, then 1 is the *only* idempotent: for if $e^2 = e$ then $ee = e1$ and so $e = 1$ by cancellation. In particular for S a group we have $E(S) = \{1\}$.