Definition 2.3. Let S, T be semigroups then $\theta : S \to T$ is a semigroup (homo)*morphism* if, for all $a, b \in S$,

$$(ab)\theta = a\theta b\theta.$$

If S, T are monoids then θ is a monoid (homo)*morphism* if θ is a semigroup morphism and $1_S\theta = 1_T$.

Example 2.4. (1) $\theta : B \to \mathbb{Z}$ given by $(a, b)\theta = a - b$ is a monoid morphism

$$\begin{aligned} ((a,b)(c,d))\theta &= (a-b+t, d-c+t)\theta \qquad t = \max\{b, c\} \\ &= (a-b+t) - (d-c+t) \\ &= (a-b) + (c-d) \\ &= (a,b)\theta + (c,d)\theta. \end{aligned}$$

Furthermore $(0,0)\theta = 0 - 0 = 0$.
(2) Let $T = I \times J$ be the rectangular band then define $\alpha : T \to \mathcal{T}_J$ by $(i,j)\alpha = c_j$. Then we have

$$\begin{aligned} ((i,j)(k,\ell))\alpha &= (i,\ell)\alpha, \\ &= c_\ell, \\ &= c_j c_\ell, \\ &= (i,j)\alpha(k,\ell)\alpha. \end{aligned}$$

So, α is a morphism.

Definition 2.5. A bijective morphism is an *isomorphism*.

Isomorphisms preserve algebraic properties (e.g. commutativity).

See handout for further information.

Embeddings Suppose $\alpha : S \to T$ is a morphism. Then Im α is a subsemigroup (submonoid) of T. If α is 1:1, then $\alpha : S \to$ Im α is an isomorphism, so that $S \cong$ Im α. We say that S is *embedded* in T.