$$\begin{aligned}\cancel{a-b} - d + c + \max\big\{d - c + \max\{b,c\}, e\big\} &= \cancel{a-b} + \max\big\{b, c - d + \max\{d,e\}\big\},\\ \cancel{f-e} + \max\big\{d - c + \max\{b,c\}, e\big\} &= \cancel{f-e} - c + d + \max\big\{b, c - d + \max\{d,e\}\big\}.\end{aligned}$$

We can see that these equations are the same and so we only need to show

$$c - d + \max\big\{d - c + \max\{b,c\}, e\big\} = \max\big\{b, c - d + \max\{d,e\}\big\}.$$

Now, we have from ($\star$) that this is equivalent to

$$\max\big\{\max\{b,c\}, c - d + e\big\} = \max\big\{b, c - d + \max\{d,e\}\big\}.$$

The RHS of this equation is

$$\begin{aligned}\max\big\{b, c - d + \max\{d,e\}\big\} &= \max\big\{b, \max\{c - d + d, c - d + e\}\big\},\\ &= \max\big\{b, \max\{c, c - d + e\}\big\},\\ &= \max\{b, c, c - d + e\},\\ &= \max\big\{\max\{b,c\}, c - d + e\big\}.\end{aligned}$$

Therefore multiplication is associative and hence B is a monoid. □

Definition 1.11. With the above multiplication, B is called the *Bicyclic Semigroup/Monoid*.

Example 1.12. For any set X, the set $\mathcal{T}_X$ of all maps $X \to X$ is a monoid. (See Lecture 3).

Definition 1.13. A semigroup S is *commutative* if $ab = ba$ for all $a, b \in S$.

For example $\mathbb{N}$ with $+$ is commutative. B is not because

$$\begin{aligned}(0,1)(1,0) &= (0 - 1 + 1, 0 - 1 + 1) = (0,0),\\ (1,0)(0,1) &= (1 - 0 + 0, 1 - 0 + 0) = (1,1).\end{aligned}$$

Thus we have $(0,1)(1,0) \neq (1,0)(0,1)$. Notice that in B; $(a,b)(b,c) = (a,c)$.

Definition 1.14. A semigroup is *cancellative* if

$$\begin{aligned}ac = bc &\Rightarrow a = b, \text{ and}\\ ca = cb &\Rightarrow a = b.\end{aligned}$$

NOT ALL SEMIGROUPS ARE CANCELLATIVE

For example in the rectangular band on $\{1,2\} \times \{1,2\}$ we have

$$(1,1)(1,2) = (1,2) = (1,2)(1,2)$$