- $\max\{a, 0\} = a$ if $a \in \mathbb{N}^0$,
- $\max\{a, b\} = \max\{b, a\}$,
- $\max\{a, a\} = a$,
- $\max\big\{a, \max\{b, c\}\big\} = \max\{a, b, c\} = \max\big\{\max\{a, b\}, c\big\}$.

Thus we have that $(\mathbb{Z}, \max)$ where $\max(a, b) = \max\{a, b\}$ is a semigroup and $(\mathbb{N}^0, \max)$ is a monoid.

NOTE. The following identities hold for all $a, b, c \in \mathbb{Z}$

$$(\star)\begin{cases} a + \max\{b, c\} = \max\{a + b, a + c\}, \\ \quad\max\{b, c\} = a + \max\{b - a, c - a\}. \end{cases}$$

Put $B = \mathbb{N}^0 \times \mathbb{N}^0$. On B we define a 'binary operation' by

$$(a, b)(c, d) = (a - b + t, d - c + t),$$

where $t = \max\{b, c\}$.

Proposition 1.10. *B is a monoid with identity* $(0, 0)$.

Proof. With $(a, b), (c, d) \in B$ and $t = \max\{b, c\}$ we have $t - b \geqslant 0$ and $t - c \geqslant 0$. Thus we have $a - b + t \geqslant a$ and $d - c + t \geqslant d$. Therefore, in particular $(a - b + t, d - c + t) \in B$ so multiplication is closed. We have that $(0, 0) \in B$ and for any $(a, b) \in B$ we have

$$\begin{aligned}(0, 0)(a, b) &= (0 - 0 + \max\{0, a\}, b - a + \max\{0, a\}),\\ &= (0 - 0 + a, b - a + a),\\ &= (a, b),\\ &= (a, b)(0, 0).\end{aligned}$$

Therefore $(0, 0)$ is the identity of B.
We need to verify associativity.
Let (a, b), (c, d), $(e, f) \in B$. Then

$$\begin{aligned}\big((a, b)(c, d)\big)(e, f) &= \big(a - b + \max\{b, c\}, d - c + \max\{b, c\}\big)(e, f),\\ &= \big(a - b - d + c + \max\{d - c + \max\{b, c\}, e\},\\ &\qquad f - e + \max\{d - c + \max\{b, c\}, e\}\big).\\ (a, b)\big((c, d)(e, f)\big) &= (a, b)\big(c - d + \max\{d, e\}, f - e + \max\{d, e\}\big),\\ &= \big(a - b + \max\{b, c - d + \max\{d, e\}\}\\ &\qquad f - e - c + d + \max\{b, c - d + \max\{d, e\}\}\big).\end{aligned}$$

Now we have to show that