NOTE. The identity of a monoid is unique.

DEFINITION 1.3. A *group* G is a monoid such that for all $a \in G$ there exists a $b \in G$ with $ab = 1 = ba$.

EXAMPLE 1.4. Groups are monoids and monoids are semigroups. Thus we have

$$\text{Groups} \subset \text{Monoids} \subset \text{Semigroups}.$$

The one element trivial group $\{e\}$ with multiplication table

	e
e	e

is also called the *trivial semigroup* or *trivial monoid*.

EXAMPLE 1.5. A ring is a semigroup under $\times$. If the ring has an identity then this semigroup is a monoid.

EXAMPLE 1.6. (1) $(\mathbb{N}, \times)$ is a monoid.
(2) $(\mathbb{N}, +)$ is a semigroup.
(3) $(\mathbb{N}^0, \times)$ and $(\mathbb{N}^0, +)$ are monoids.

EXAMPLE 1.7. Let I, J be non-empty sets and set $T = I \times J$ with the binary operation

$$(i, j)(k, \ell) = (i, \ell).$$

Note

$$\big((i,j)(k,\ell)\big)(m,n) = (i,\ell)(m,n) = (i,n),$$
$$(i,j)\big((k,\ell)(m,n)\big) = (i,j)(k,n) = (i,n),$$

for all $(i,j), (k,\ell), (m,n) \in T$ and hence multiplication is associative.
Then T is a semigroup called the *rectangular band* on $I \times J$.

Notice: $(i,j)^2 = (i,j)(i,j) = (i,j)$, i.e. every element is an idempotent.

This shows that not every semigroup is the multiplicative semigroup of a ring, since any ring where every element is an idempotent is commutative. However, a rectangular band does not *have to be commutative.*

Adjoining an Identity Let S be a semigroup. Find a symbol not in S, call it "1". On $S \cup \{1\}$ we define $*$ by

$$\begin{aligned} a * b &= ab && \text{for all } a, b \in S,\\ a * 1 = a &= 1 * a && \text{for all } a \in S,\\ 1 * 1 &= 1. \end{aligned}$$