SEMIGROUP THEORY
A LECTURE COURSE

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1. The Basic Concept

Definition 1.1. A *semigroup* is a pair $(S, *)$ where S is a non-empty set and $*$ is an associative binary operation on S. [i.e. $*$ is a function $S \times S \to S$ with $(a, b) \mapsto a * b$ and for all $a, b, c \in S$ we have $a * (b * c) = (a * b) * c$].

n	Semigroups	Groups
1	1	1
2	4	1
3	18	1
4	126	2
5	1160	1
6	15973	2
7	836021	1
8	1843120128	5
9	52989400714478	2

The number (whatever it means) of semigroups and groups of order n

We abbreviate "$(S, *)$" by "S" and often omit $*$ in "$a * b$" and write "ab". By induction $a_1 a_2 \dots a_n$ is unambiguous. Thus we write a^n for

$$\underbrace{aa\dots a}_{n \text{ times}}.$$

Index Laws For all $n, m \in \mathbb{N} = \{1, 2, \dots\}$:

$$a^n a^m = a^{n+m}$$
$$(a^n)^m = a^{nm}.$$

Definition 1.2. A *monoid* M is a semigroup with an identity, i.e. there exists $1 \in M$ such that $1a = a = a1$ for all $a \in M$.

Putting $a^0 = 1$ then the index laws hold for all $n, m \in \mathbb{N}^0 = \{0, 1, 2, \dots\}$.