Volterra integral equations

2.1 The method of successive approximations:

In this method, we replace the unknown function
$u(x) = u_0(x)$, (zero approximation)
This substitution will give the first approximation $u_1(x)$ by
$u_1(x) = f(x) + \lambda \int_0^x k(x,t)u_0(t)\,dt$, (first approximation)
It is obvious that $u_1(x)$ is continuous if $f(x), k(x,t)$ & $u_0(x)$ are continuous.
$u_2(x) = f(x) + \lambda \int_0^x k(x,t)u_1(x)\,dt$ (second approximation)
Continuing in this manner
$u_n(x) = f(x) + \lambda \int_0^x k(x,t)u_{n-1}(x)\,dt \quad for\ n = 1,2,3,\dots$
$u(x) = \lim_{n\to\infty} u_n(x)$
Now if we choose $u_0(x) = f(x)$ then it is called Picard's successive approximation method

$$u_1(x) = f(x) + \lambda \int_0^x k(x,t)f(t)\,dt,$$
$$u_2(x) = f(x) + \lambda \int_0^x k(x,t)u_1(t)\,dt,$$
$$\vdots$$
$$u_n(x) = f(x) + \lambda \int_0^x k(x,t)u_{n-1}(t)\,dt,$$

Consider

$$u_2(x) - u_1(x) =$$
$$\lambda \int_0^x k(x,t)\left[f(x) + \lambda \int_0^t k(x,\tau)f(\tau)\,d\tau\right]dt - \lambda \int_0^x k(x,t)f(t)\,dx$$
$$= \lambda^2 \int_0^x k(x,t) \int_0^t k(x,\tau)f(\tau)\,d\tau\,dt$$
$$= \lambda^2 \psi_2(x)$$

Where $\psi_2(x) = \int_0^x k(x,t)dt \int_0^t k(x,\tau)f(\tau)\,d\tau$ **1**
So it can easily seen that
$u_n(x) = \sum_{m=0}^{n} \lambda^m \psi_m(x) \quad if\ \psi_0(x) = f(x)$
Then $\psi_m(x) = \int_0^x k(x,t)\psi_{m-1}(t)dt, \quad m = 1,2,3,\dots$
Where $\psi_1(x) = \int_0^x k(x,t)f(t)dt$

The repeated integrals in equation **1** may be considered as a double integral over the triangular region indicated in Figure 3, thus interchanging the order of integration

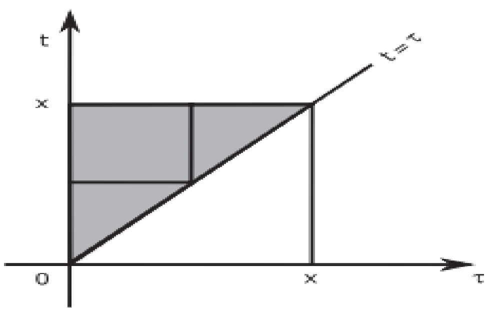

Figure 2.1: Double integration over the triangular region (shaded area).

$\psi_2(x) = \int_0^x f(\tau)d\tau \int_\tau^x k(x,t)k(t,\tau)\,dt$

$\qquad = \int_0^x k_2(x,\tau)f(\tau)d\tau \quad where\ k_2(x,\tau) = \int_\tau^x k(x,t)k(t,\tau)\,dt$

$\psi_m(x) = \int_0^x k_m(x,\tau)f(\tau)d\tau\,,\ m = 1,2,3,\dots$

Where $\psi_{m+1}(x) = \int_0^x k(x,\tau)k_m(\tau,t)d\tau$

$u_n(x) = f(x) + \sum_{m=1}^n \lambda^m \psi_m(x)$

Also $u_n(x) = f(x) + \int_0^x [\sum_{m=1}^n \lambda^m k_m(x,\tau)]\, f(\tau)d\tau$

$\lim_{n\to\infty} u_n(x) = u(x)$

$\lim_{n\to\infty} u_n(x) = f(x) + \lambda \int_0^x H(x,\tau;\lambda)\,d\tau \quad where\ \ H(x,\tau;\lambda) = \lambda^m k_m(x,\tau)$

$H(x,\tau;\lambda)$ is known as the resolvent kernel.

Examples:

Example 1:

Solve the Volterra integral equation $u(x) = x + \int_0^x (t-x)\,u(t)\,dt.$

Solution:

$\because f(x) = x, \lambda = 1\ \&\ k(x,t) = t - x$

$u_0(x) = x$

$u_1(x) = x + \int_0^x (t-x)\,u_0(t)\,dt \Rightarrow x + \int_0^x (t-x)\,t\,dt$

$\Rightarrow x + \int_0^x (t^2 - xt)\ dt \Rightarrow x + \left[\frac{1}{3}t^3 - \frac{1}{2}xt^2\right]_0^x \Rightarrow x + \frac{1}{3}x^3 - \frac{1}{2}x^3$

$\Rightarrow u_1(x) = x - \frac{1}{6}x^3 \Rightarrow u_1(x) = x - \frac{1}{3!}x^3$

$u_2(x) = x + \int_0^x (t-x)\,u_1(t)\,dt \Rightarrow x + \int_0^x (t-x)\left(t - \frac{1}{6}t^3\right)dt$

$\Rightarrow u_2(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \Rightarrow u_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

$u_3(x) =$

$\therefore n - th$ approximation is given by

$u_n(x) = \sum_{m=1}^n (-1)^{m-1} \frac{x^{2m-1}}{(2m-1)!} \quad n \geq 1$

$$\Rightarrow u(x) = \lim_{n\to\infty} u_n(x) \Rightarrow \lim_{n\to\infty}\left((-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}\right) = sin(x)$$

$\therefore\ u(x) = sin(x)$

Example 2:

Solve the Volterra integral equation $u(x) = 1 - \int_0^x (t - x)\, u(t)\, dt.$

Solution:

$\because f(x) = 1, \lambda = 1$ & $k(x,t) = t - x$

$u_0(x) = 1$

$u_1(x) = 1 - \int_0^x (t - x)\, u_0(t)\, dt \Rightarrow 1 - \int_0^x (t - x)\, 1\, dt$

$\Rightarrow 1 - \int_0^x (t - x)\ dt \Rightarrow 1 - \left[\frac{1}{2}t^2 - xt\right]_0^x \Rightarrow 1 - \frac{1}{2}x^2 + x^2$

$\Rightarrow u_1(x) = 1 + \frac{1}{2}x^2 \Rightarrow u_1(x) = 1 + \frac{1}{2!}x^2$

$u_2(x) = 1 - \int_0^x (t - x)\, u_1(t)\, dt \Rightarrow 1 - \int_0^x (t - x)\left(1 + \frac{1}{2}t^2\right) dt$

$\Rightarrow u_2(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 \Rightarrow u_2(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4$

$u_3(x) =$

$\therefore n - th$ approximation is given by

$u_n(x) = \sum_{m=1}^{n} \frac{x^{2(m-1)}}{2(m-1)!} \quad n \geq 1 \quad \Rightarrow \ u_n(x) = \sum_{m=0}^{n} \frac{x^{2m}}{2m!}$

$$\Rightarrow u(x) = \lim_{n\to\infty} u_n(x) \Rightarrow \lim_{n\to\infty}\left(\frac{x^{2(n-1)}}{2(n-1)!}\right) = cosh(x)$$

$\therefore\ u(x) = cosh(x)$