0.1 Bernoulli Distribution

This distribution describes a natural phenomenon or a mechanical process in which you expect a particular event to appear or not.

If the outcome of the random experiment is either a success with a fixed probability p or a failure with a probability $q = 1 - p$ and that the random variable X takes either the value 1 in the case of success or a value of zero in the case of failure, then the distribution of X is the Bernoulli distribution.

That is, $X = \begin{cases} 1 & \text{when the event appears} \\ 0 & \text{when the event does not appear} \end{cases}$

and

$$f(x) = \begin{cases} p & x = 1 \\ 1 - p = q & x = 0 \end{cases}$$

This function can be written in another form (Bernoulli distribution).

$$\boxed{\begin{array}{l} f(x) = P(X = x) = \begin{cases} p^x q^{1-x} & x = 0, 1, \quad 0 < p < 1, \quad q = 1 - p \\ 0 & o.w. \end{cases} \\ \text{Where } X \sim Ber(p);\ \ p \text{ is the parameter of the distribution.} \end{array}}$$

Exercise 1 *:*

1. *Prove that the Bernoulli distribution is a probability mass function.*
2. *Find the*
   - *Average.*
   - *Variance.*
   - *Moment generating function.*
   - *Cumulative distribution function.*

Solution:

1. To prove that the Bernoulli distribution is a probability mass function.

   Since $0 < p, q < 1$ then $0 < p^x q^{1-x} < 1$, which yields $0 < f(x) < 1$ when $x = 0, 1$ and since $f(x) = 0$ for otherwise ($x \neq 0, 1$). Then $0 \leq f(x) < 1$.

   Now we want to prove $\sum\limits_X f(x) = 1$.

   $$\sum_X f(x) = \sum_{X=0}^{1} p^X q^{1-X} = p^0 q^1 + p^1 q^0 = q + p = 1 - p + p = 1$$

   Hence, f is a p.m.f of a r.v. X.

2. To find the Average, Variance, Moment generating function for this distribution and Cumulative distribution function.
   - $\mu = E(X) = \sum\limits_X X f(x) = \sum\limits_{X=0}^{1} X \cdot p^X q^{1-X}$

     $= 0 \cdot p^0 q^1 + 1 \cdot p^1 q^0 = p$

     $\boxed{\mu = E(X) = p}$

   - $\sigma^2 X = E(X^2) - [E(X)]^2$.

     $$E(X^2) = \sum_{X=0}^{1} X^2 \cdot p^X q^{1-X} = 0 \cdot p^0 q^1 + 1 \cdot p^1 q^0 = p$$

$$\boxed{\begin{array}{l}\text{Note that the moments for any } r \text{ is } p \text{ since}\\ \mu_r = E(X^r) = \sum_{X=0}^{1} X^r \cdot p^X q^{1-X} = 0 \cdot p^0 q^1 + 1 \cdot p^1 q^0 = p\end{array}}$$

$\sigma^2 X = E(X^2) - [E(X)]^2 = p - p^2 = p(1-p) = pq.$

$\boxed{\sigma^2 X = pq.}$

- $$\begin{aligned} M_X(t) = E(e^{tX}) = \sum_X e^{tX} \cdot f(x) \\ &= \sum_{X=0}^{1} e^{tX} \cdot p^X q^{1-X} \\ &= e^0 \cdot p^0 q^1 + e^t \cdot p^1 q^0 = pe^t + q. \end{aligned}$$

  $\boxed{M_X(t) = pe^t + q}$

- C.D.F of X is

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ q & 0 \leq x < 1 \\ \\ 1 & x \geq 1 \\ \end{cases}$$

Example 1 *One dice was thrown. Let the random variable X be the number 6 shown by the dice face. Find:*

1. *p.m.f of X.*
2. *c.d.f of X.*
3. *$M_X(t)$.*
4. *$E(X)$.*
5. *$V(X)$.*
6. *$P(-3 \leq X < 0)$.*
7. *$P(-2 \leq X < 1)$.*
8. *$P(X \geq 3)$.*

9. $P(0 \leq X < 2)$.

Solution:

If the number 6 appears, then $X = 1$ (success).

If the number $1, 2, 3, 4$, or 5 appear, then $X = 0$ (failure).

Hence, $X \sim Ber(p)$ with $p = \frac{1}{6}$ and $q = 1 - p = \frac{5}{6}$

$$f(x) = P(X = x) = \begin{cases} (\frac{1}{6})^x (\frac{5}{6})^{1-x} & x = 0, 1 \\ 0 & o.w. \end{cases}$$

- C.D.F of X is

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ \frac{5}{6} & 0 \leq x < 1 \\ \\ 1 & x \geq 1 \end{cases}$$

- $M_X(t) = pe^t + q = \frac{1}{6}e^t + \frac{5}{6} = \frac{5+e^t}{6}$.

- $E(X) = p = \frac{1}{6}$.

- $V(X) = \sigma^2 X = E(X^2) - [E(X)]^2 = p \cdot q = \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$.

- $P(-3 \leq X < 0) = 0$.

- $P(0 \leq X < 2) = p(X = 0) + p(X = 1) = \frac{5}{6} + \frac{1}{6} = 1$.

- $P(-2 \leq X < 1) = p(X = -2) + p(X = -1) + p(X = 0)$
  $= 0 + 0 + \frac{5}{6} = \frac{5}{6}$.

- $P(X \geq 3) = 0$.