# Lecture (11)

# Semi-Lagrangian Advection Scheme (Part2)

## 11.1 Numerical Domain of Dependence

For the Eulerian Leapfrog Scheme, the value $Y_{p,q}$ at time $k\Delta t$ and position $p\Delta x$ depends on values within the area depicted by asterisks (See Fig 11.1).

Values outside this region have no influence on $Y_{p,q}$.

Each computed value $\mathbf{Y_{p,q}}$ depends on previously computed values and on the initial conditions. The set of points which influence the value $Y_{p,q}$ is called the numerical domain of dependence of $Y_{p,q}$.

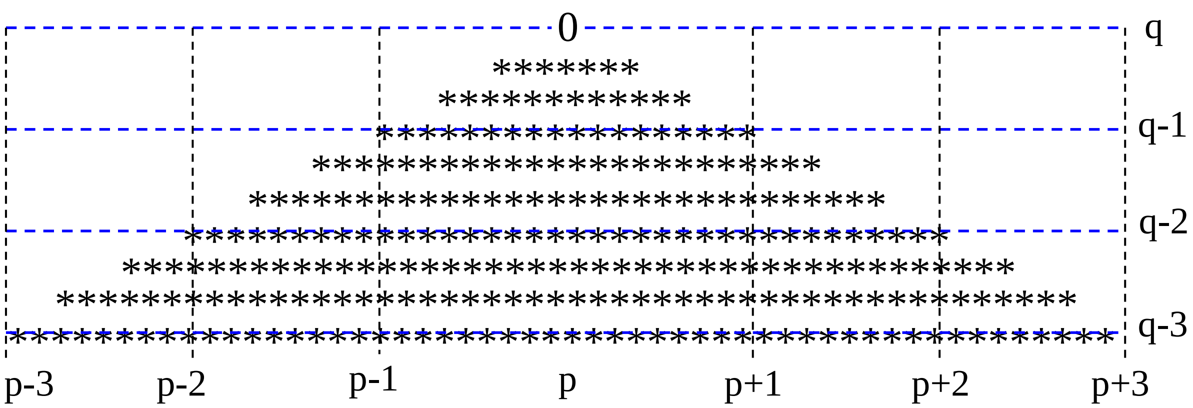


**Figure 11.1 Numerical domain of dependence**

It is clear on physical grounds that if the parcel of fluid arriving at point $p\Delta x$ at time $q\Delta t$ originates outside the numerical domain of dependence, the numerical scheme cannot yield an accurate result: the necessary information is not available to the scheme.

Worse again, the numerical solution may bear absolutely no relationship to the physical solution and may grow exponentially with time even when the true solution is bounded.

A necessary condition for avoidance of this phenomenon is that the numerical domain of dependence should include the physical trajectory. This condition is fulfilled by the semi-Lagrangian scheme.

## 11.2 Parcel coming from outside domain of dependence

The line of bullets (•) represents a parcel trajectory. The value everywhere on the trajectory is $Y_{p,q}$ (See Figure 11.2).

Since the parcel originates outside the numerical domain of dependence, the Eulerian scheme cannot model it correctly. The central idea of the Lagrangian scheme is to represent the physical trajectory of the fluid parcel.

We consider a parcel arriving at gridpoint mΔx at the new time $(q+1)\Delta t$ and ask: Where has it come from?

The departure point will not normally be a grid point. Therefore, the value at the departure point must be calculated by interpolation from surrounding points. But this interpolation ensures that the trajectory falls within the numerical domain of dependence. We will show that this leads to a numerically stable scheme.

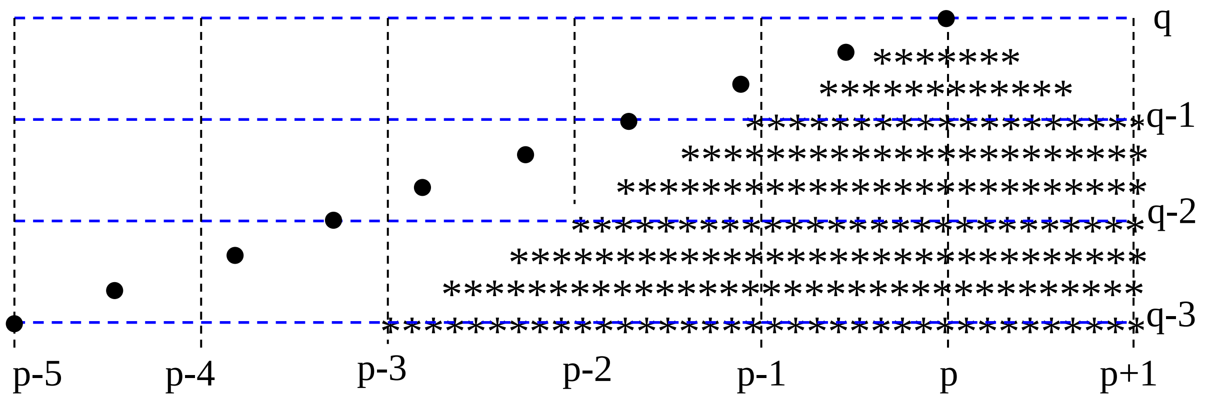


**Figure 14.2 Parcel trajectory**

## 11.3 Interpolation Using Surrounding Points

The line of circles (◦) represents a parcel trajectory ($c = \frac{5\Delta x}{3\Delta t}$) At time $(q-1)\Delta t$ the parcel is at (•), which is not a grid-point (See Figure 11.3).

The value at the departure point is obtained by interpolation from surrounding points. Thus we ensure that, even though $CFL = \frac{5}{3} > 1$, the physical trajectory is within the domain of numerical dependence.

The advection equation in Lagrangian form may be written $\frac{dY}{dt} = 0$.

From a physical aspect, this equation says that the value of Y is constant for a fluid parcel. Applying the equation over the time interval $[q\Delta t, (q+1)\Delta t]$, we get

$$\begin{pmatrix} Value\ of\ Y\ at\ point \\ p\Delta x\ at\ time\ (q+1)\Delta t \end{pmatrix} = \begin{pmatrix} Value\ of\ Y\ at\ departure \\ point\ at\ time\ q\Delta t \end{pmatrix}$$

In a more compact form, we may write $Y_{p,q+1} = Y_{\bullet,q}$ where $Y_{\bullet,q}$ represents the value at the departure point, which is normally not a grid point.

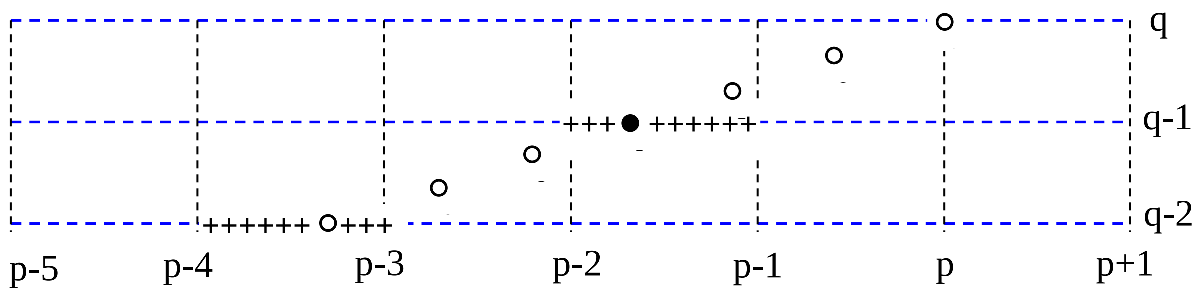


**Figure 11.3 Parcel trajectory**

The distance travelled in time $\Delta t$ is $s = c\Delta t$.
We define the integer and fractional parts of $s$ as follows:
$\gamma = Integer\ part\ of\ s$
$\alpha = s - \gamma = Fractional\ part\ of\ s$
Note that, by definition, $0 \leq \alpha < 1$.
So, the departure point falls between the grid points $p - \gamma - 1$ and $p - \gamma$. In the figure (11.4), $\gamma = 1$ and $\alpha \approx 2/3$. A linear interpolation gives:

$$Y_{\bullet,q} = \alpha Y_{p-\gamma-1,q} + (1-\alpha) Y_{p-\gamma,q}$$

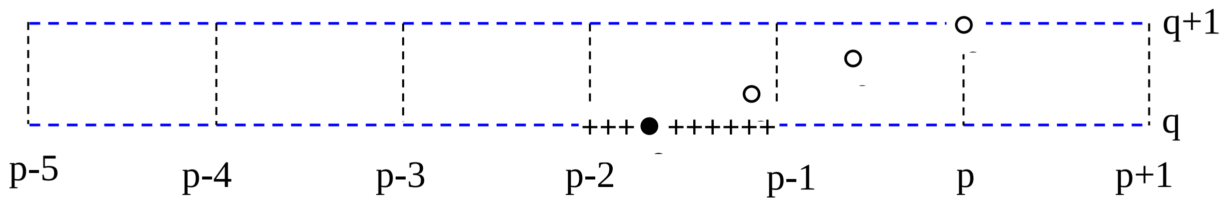


**Figure 11.4 Parcel trajectory**

**11.4 Numerical Stability of the Scheme**

The discrete equation may be written

$Y_{p,q+1} = \alpha Y_{p-\gamma-1,q} + (1-\alpha)Y_{p-\gamma,q}$

Let us look for a solution of the form

$Y_{p,q} = \mathrm{a}A_q exp(ikp\Delta x)$.

Substituting into the equation, we get

$\mathrm{a}A_{q+1}exp(ikp\Delta x) = \mathrm{a}\alpha A_q \exp[ik(p-\gamma-1)\Delta x] + (1-\alpha)\mathrm{a}A_q \exp[ik(p-\gamma)\Delta x]$

Removing the common term $\mathrm{a}A_q exp(ikp\Delta x)$, we get

$A = \alpha \exp[ik(-\gamma-1)\Delta x)] + (1-\alpha)\exp[ik(-\gamma)\Delta x]$

We can write this as

$A = \alpha \exp(-ik\gamma\Delta x). \exp(-ik\Delta x) + (1-\alpha)\exp(-ik\gamma\Delta x)$

$A = \exp(-ik\gamma\Delta x) \cdot [(1-\alpha) + \alpha \exp(-ik\Delta x)]$

Now consider the squared modulus of A:

$|A|^2 = |exp(-ik\gamma\Delta x)|^2 . |(1-\alpha) + \alpha\, exp(-ik\Delta x)|^2$

$= |\,(1-\alpha) + \alpha\, cos\, k\Delta x - i\alpha\, sin\, k\Delta x|^2$

$= [\,(1-\alpha) + \alpha\, cos\, k\Delta x]^2 + [\alpha\, sin\, k\Delta x]^2$

$= (1-\alpha)^2 + 2(1-\alpha)\alpha\, cos\, k\Delta x + \alpha^2\, cos^2\, k\Delta x + \alpha^2\, sin^2\, k\Delta x$

$= 1 - 2\alpha + \alpha^2 + 2\alpha\, cos\, k\Delta x - 2\alpha^2\, cos\, k\Delta x + \alpha^2$

$= 1 - 2\alpha + 2\alpha^2 + 2\alpha\, cos\, k\Delta x - 2\alpha^2\, cos\, k\Delta x$

$= 1 - 2\alpha + 2\alpha^2 + 2\alpha\, cos\, k\Delta x\, (1-\alpha)$

$= 1 - 2\alpha(1-\alpha) + 2\alpha\, cos\, k\Delta x\, (1-\alpha) \quad = 1 - 2\alpha(1-\alpha)[1 - \, cos\, k\Delta x]$

We note that $0 \le (1 - cos\, k\Delta x) \le 2$

Taking the largest value of $1 - cos\, k\Delta x$ (i.e. 2) gives:

$|A|^2 = 1 - 4\alpha(1-\alpha) = (1 - \, 2\alpha)^2 < 1 \quad because\ \ \alpha < 1$

Taking the smallest value of $1 - \, cos\, k\Delta x$ gives $|A|^2 = 1$

In either case, $|A|^2 = 1$ , so there is numerical stability.