Lecture (3)

Finite Difference Methods (Part 1)

The finite difference method (FDM) is an approximate method for solving partial differential equations. It has been used to solve a wide range of problems. These include linear and non-linear, time independent and dependent problems. This method can be applied to problems with different boundary shapes, different kinds of boundary conditions, and for a region containing a number of different materials.

The application of FDM is not difficult as it involves only simple arithmetic in the derivation of the discretization equations and in writing the corresponding programs.

3.1 One dimension finite difference method

Consider the derivative $\frac{df}{dx}$, where $f = f(x)$, and x is the independent variable (which could be either space or time). Finite-difference methods represent the continuous function $f(x)$ by a set of values defined at a finite number of discrete points in a specified (spatial or temporal) region. Thus, we usually introduce a "grid" with discrete points where the variable f is defined,

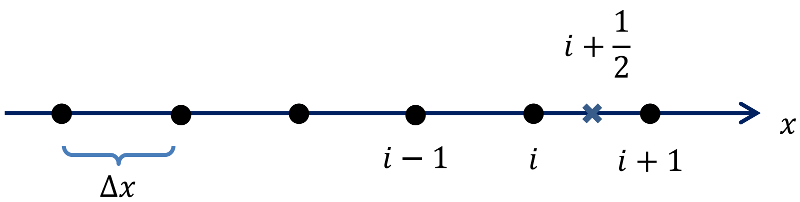

Figure 3.1 An example of a grid, with uniform spacing Δx (grid size).

We assume that the grid spacing is constant, then $x_i = i\Delta x$, where i is the index used to identify the grid points. Using the notation $f_i = f(x_i) = f(i\Delta x)$, we can define

The *forward difference* at the point i by $f_{i+1} - f_i$, (3.1)

The *backward difference* at the point i by $f_i - f_{i-1}$, (3.2)

The *centered difference* quotient at the point $i + \frac{1}{2}$ by $f_{i+1} - f_i$, (3.3)

Note that f itself is not defined at the point $i+\frac{1}{2}$. From (3.1) - (3.3) we can define the following "finite-difference quotients:"

The forward-difference quotient at the point *i* :

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_{i+1}-f_i}{\Delta x}\ ; \qquad (3.4)$$

The backward – difference quotient at the point *i*:

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_i-f_{i-1}}{\Delta x}\ ; \qquad (3.5)$$

and the centered-difference quotient at the point $i+\frac{1}{2}$:

$$\left(\frac{df}{dx}\right)_{i+\frac{1}{2},\text{approx}} = \frac{f_{i+1}-f_i}{\Delta x}\ ; \qquad (3.6)$$

In addition, the centered-difference quotient at the point *i* can be defined by

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_{i+1}-f_{i-1}}{2\Delta x}\ ; \qquad (3.7)$$

Since (3.4) and (3.5) employ the values of f at two points, and give an approximation to $\frac{df}{dx}$ at one of the same points, they are sometimes called *two-point approximations*. On the other hand, (3.6) and (3.7) are *three-point approximations*, because the approximation to $\frac{df}{dx}$ is defined at a location different from the locations of the two values of f on the right-hand side of the equals sign.

How accurate are these finite-difference approximations?

"Accuracy" can be measured in a variety of ways, as we shall see. One measure of accuracy is *truncation error*. The term refers to the truncation of an infinite series expansion. As an example, consider the forward difference quotient

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_{i+1}-f_i}{\Delta x} \equiv \frac{f[(i+1)\Delta x]-f(i\Delta x)}{\Delta x}\ ; \qquad (3.8)$$

Expand f in a Taylor series about the point x_i , as follows:

$$f_{i+1} = f_i + \Delta x\left(\frac{df}{dx}\right)_i + \frac{(\Delta x)^2}{2!}\left(\frac{d^2f}{dx^2}\right)_i + \frac{(\Delta x)^3}{3!}\left(\frac{d^3f}{dx^3}\right)_i + \cdots + \frac{(\Delta x)^{n-1}}{(n-1)!}\left(\frac{d^{n-1}f}{dx^{n-1}}\right)_i + \cdots \qquad (3.9)$$

Eq. (3.9) can be rearranged to

$$\frac{f_{i+1} - f_i}{\Delta x} = \left(\frac{df}{dx}\right)_i + \varepsilon \qquad (3.10)$$

where

$$\varepsilon \equiv \frac{(\Delta x)^2}{2!}\left(\frac{d^2 f}{dx^2}\right)_i + \frac{(\Delta x)^3}{3!}\left(\frac{d^3 f}{dx^3}\right)_i + \cdots + \frac{(\Delta x)^{n-1}}{(n-1)!}\left(\frac{d^{n-1} f}{dx^{n-1}}\right)_i + \cdots \quad (3.11)$$

is called the *truncation error*.

If Δx is small enough, the leading term on the right-hand side of Eq (3.11) will be the largest part of the error. *The lowest power of Δx that appears in the truncation error is called the "order of accuracy" or "order of approximation" of the corresponding difference quotient*. For example, the leading term of (3.10) is of order Δx, abbreviated as O(Δx), and so we say that (3.10) is a first order approximation or an approximation of first-order accuracy. Obviously (3.5) is also first-order accurate. Just to be as clear as possible, a first-order scheme for the first derivative has the form $\left(\frac{df}{dx}\right)_{i,\text{approx}} = \left(\frac{df}{dx}\right)_i + O[\Delta x]$, where $\left(\frac{df}{dx}\right)_{i,\text{approx}}$ is an *approximation* to the first derivative and $\left(\frac{df}{dx}\right)_i$ is the true first derivative. Similarly, a second- order scheme for the first derivative has the form $\left(\frac{df}{dx}\right)_{i,\text{approx}} = \left(\frac{df}{dx}\right)_i + O[(\Delta x)^2]$, and so on for higher orders of accuracy.

Similar analyses of (3.6) and (3.7) show that they are of second-order accuracy. For example, we can write

$$f_{i-1} = f_i + \left(\frac{df}{dx}\right)_i(-\Delta x) + \left(\frac{d^2 f}{dx^2}\right)_i \frac{(-\Delta x)^2}{2!} + \left(\frac{d^3 f}{dx^3}\right)_i \left[\frac{-(\Delta x)^3}{3!}\right] + \cdots \qquad (3.12)$$

Subtracting (3.12) from (3.9) gives

$$f_{i+1} - f_{i-1} = 2\left(\frac{df}{dx}\right)_i(\Delta x) + \frac{2}{3!}\left(\frac{d^3 f}{dx^3}\right)_i[(\Delta x)^3] + \cdots \quad \text{odd powers only}, \qquad (3.13)$$

which can be rearranged to

$$\left(\frac{df}{dx}\right)_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \left(\frac{d^3 f}{dx^3}\right)_i \frac{\Delta x^2}{3!} + O[(\Delta x)^4] \qquad (3.14)$$

Similarly,

$$\left(\frac{df}{dx}\right)_{i+\frac{1}{2}} \cong \frac{f_{i+1}-f_i}{\Delta x} - \left(\frac{d^3 f}{dx^3}\right)_{i+\frac{1}{2}} \frac{(\Delta x/2)^2}{3!} + O[(\Delta x)^4]. \qquad (3.15)$$

From (3.14) and (3.15), we see that

$$\left|\frac{Error\ of\ (3.14)}{Error\ of\ (3.15)}\right| \cong \frac{\left(\frac{d^3 f}{dx^3}\right)_i \frac{\Delta x^2}{3!}}{\left(\frac{d^3 f}{dx^3}\right)_{i+\frac{1}{2}} \frac{(\Delta x/2)^2}{3!}} = \frac{4\left(\frac{d^3 f}{dx^3}\right)_i}{\left(\frac{d^3 f}{dx^3}\right)_{i+\frac{1}{2}}} \cong 4 \qquad (3.16)$$

This shows that the error of (3.14) is about four times as large as the error of (3.15), even though both finite-difference quotients have second-order accuracy. The point is that the *"order of accuracy" tells how rapidly the error changes as the grid is refined, but it does not tell how large the error is for a given grid size*. It is possible for a scheme of low-order accuracy to give a more accurate result than a scheme of higher-order accuracy, if a finer grid spacing is used with the low-order scheme.

Exercises

Q1. Define: Finite difference Method, Truncation error, Order of accuracy.

Q2. (IMP.) Prove that the error in centered-difference quotient at i is four times the error in the centered-difference quotient at the point $i + \frac{1}{2}$.

MATLAB Work

Write Matlab script to calculate the air pressure for the following temperatures (20, 15, 10, 5, 0, -5, -10, -20, -40, -60) °C by using iteration loop, then draw the results. (Note: Use ideal gas law in Lecture 1).

Homework

1. The centered-difference quotient (Eq 3.7) is more accurate than the forward- and backward-difference quotients (Eq 3.4 and 3.5, respectively). Prove that.
2. Are the equations (3.4-3.16) ordinary differential equations or partial differential equations and why?