Lecture (2)

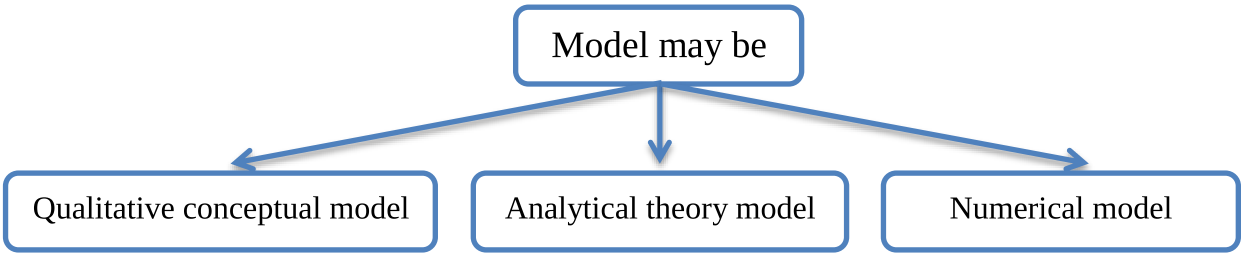
Some Introductory Principles

2.1 What is a model?

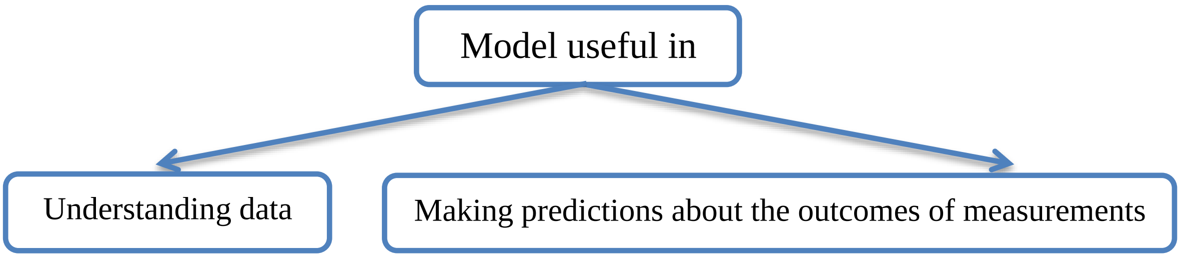
The atmospheric science researchers interest and carry out measurements of the atmosphere. They may do the following:

- Instrument development,
- algorithm development,
- data collection,
- data reduction,
- data analysis.

The data by themselves are just numbers. In order to make physical sense of the data, some sort of model is needed.



And,



Most models in atmospheric sciences are formulated by starting from basic physical principles, such as conservation of mass, conservation of momentum, and conservation of thermodynamic energy. Many of these equations are <u>prognostic</u>, which means that they involve time derivatives. A simple example is the continuity equation, which expresses conservation of mass:

$$\frac{\partial \rho}{\partial t} = -\nabla.\left(\rho \vec{V}\right) \qquad (2.1)$$

Here t is time, ρ is density, $\vec{V}$ is velocity vector. This equation is prognostic in which ρ is a prognostic variable.

A model that contains prognostic equations is solved by time integration, and the prognostic variables of such a model must be assigned initial conditions. Any variable that is not prognostic is called *diagnostic*.

2.2 Elementary models (Analytical Models)

These are the models that are essentially direct applications of the physical principles (such as conservation of mass, conservation of momentum, and conservation of thermodynamic energy) to phenomena that occur in the atmosphere. These models are "elementary" in the sense that they form the conceptual foundation for other modeling work. Elementary models are usually *analytical*. This means that the results that they produce consist of equations. As a simple example, consider the ideal gas law

$$p = \rho RT \qquad (2.2)$$

Eq. (2.2) can be derived using the kinetic theory of gases, which is an analytical model; the ideal gas law can be called a "result" of the model. This simple formula can be used to generate numbers, of course; for example, given the density and temperature of the air, and the gas constant, we can use Equation (2) to compute the pressure. This particular formula is sufficiently simple that we can understand what it means just by looking at it.

2.3 Numerical models

The results of a numerical model consist of numbers, which represent particular "cases". For example, we can "run" a numerical model to create a weather forecast. The forecast consists of a large set of numbers. To perform a new forecast, for a different initial condition, we have to run the model again, generating a new set of numbers. In order to see everything that the (model) atmosphere can do, we would have to run the model for infinitely many cases. In this way, numerical models are quite different from analytical models, which can describe all possibilities in a single formula. We cannot understand the results of a numerical simulation just by looking at the computer code.

Simply, the numerical model can be defined: is a set of discredited mathematical equations, solved on a computer, which represent the behavior of a physical system.

Question: What is the difference between analytical and numerical methods?

Ans: Analytical is exact; numerical is approximate. For example, some differential equations cannot be solved exactly (analytic or closed form solution) and we must rely on numerical techniques to solve them.

2.4 Physical and mathematical errors

All models contain errors. It is useful to distinguish between physical errors and mathematical errors. For example, we often consider the Navier-Stokes equations to be an exact description of the fluid dynamics of air. For various reasons, we are unable to obtain exact solutions to the Navier-Stokes equations. To simplify the problem, we introduce physical approximations. A second motivation for making physical approximations is that the approximate equations may describe the phenomena of interest more directly, omitting or "filtering" phenomena of less interest, and so yielding a set of equations that is more focused on and more appropriate for the problem at hand. For example, we may choose approximations that filter gravity waves (e.g., the quasigeostrophic approximation). Thise physical approximation introduce physical errors, which may or may not be considered acceptable for the intended application of a model.

At this point, we have chosen the equation system of the model. Once we have settled on a suitable set of physical equations, we must devise mathematical methods to solve them. The mathematical methods are almost always approximate and introduce errors.

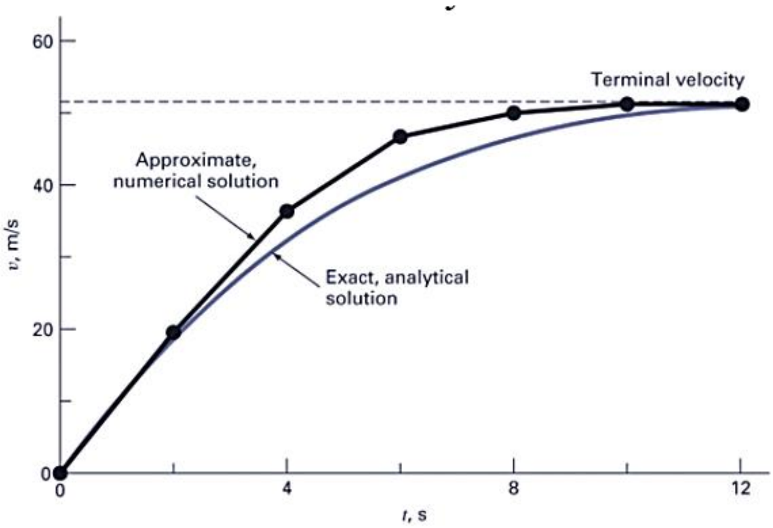
2.5 Discretization

Numerical models are "discrete." This simply means that a numerical model involves a finite number of numbers. The process of approximating a continuous model by a discrete model is called "discretization."

There are multiple approaches to discretization. This course emphasizes grid-point methods, which are sometimes called finite-difference methods. The fields of the model are defined at the discrete points of a grid. The grid can and usually does span time as well as space. Derivatives are then approximated in terms of differences involving neighboring grid-point values. A finite-difference equation (or set of equations) that approximates a differential equation (or set of equations) is called a *finite-difference scheme*, or a grid-point scheme. Grid-point schemes can be derived by various approaches, and the derivation methods themselves are sometimes given names. Examples include *finite-volume methods*, *finite-element methods*, and *semi-Lagrangian methods*.

The major alternative to the finite-difference method is the *spectral method*, which involves expanding the fields of the model in terms of weighted sums of continuous, and therefore differentiable, basis functions. Simple examples would include Fourier expansions and spherical harmonic expansions. Spectral models use grid-point methods to represent the temporal and vertical structures of the solution.

- If we choose a grid-point method, then we have to choose the *shapes* of the grid cells. Possibilities include rectangles, triangles, and hexagons.
- Having chosen the shapes of the grid cells, we must choose where to locate the predicted quantities on the grid. In many cases, different quantities will be located in different places. This is called "*staggering*." We will discuss systematic approaches to identifying the best staggering choices.
- For any given grid shape and staggering, we can make numerical schemes that are more accurate or less accurate. The meaning of accuracy will be discussed later. More accurate schemes have smaller errors, but less accurate schemes are simpler and faster.



2.6 Review of the vector mathematics

Scalars: are variables such as temperature and air pressure that have magnitude but not direction.
Vectors: are variables such as velocity that have magnitude and direction.

- The velocity vector:

$$\vec{V} = iu + jv + kw \quad (total\ vector) \qquad (2.3)$$

$$\vec{V}_h = iu + jv \quad (horizontal\ vector) \qquad (2.4)$$

And, u, v, w are the scalar components of velocity:

$$u = \frac{dx}{dt} \quad , v = \frac{dy}{dt} \quad , w = \frac{dz}{dt} \qquad (2.5)$$

The magnitude of the wind is its speed. The total and horizontal wind speeds are defined as: $|\vec{V}| = \sqrt{u^2 + v^2 + w^2}$, $|\vec{V}_h| = \sqrt{u^2 + v^2}$ (2.6)

- The dot product: It is a product of two vectors gives a scalar.

Let $\vec{A}$ and $\vec{B}$ are two vectors,

$$\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad (2.7) \quad \text{(How?)}$$

$\vec{A}.\vec{B} = |\vec{A}||\vec{B}| cos\theta$ (2.8)

Example 1.1: Let $\vec{A} = 2i - 1/2j - 3k,$ and $\vec{B} = -3i + j - 1/2k$

Find $\vec{A}.\vec{B}$ and the angle between the two vectors.

Solution: Using equation (2.7)

$\vec{A}.\vec{B} = -6 + (-1/2) + 3/2 \quad = -6.5 + 1.5 = -5$

$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{4 + 1/4 + 9} = \sqrt{13.75}$

$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{9 + 1 + 1/4} = \sqrt{10.25}$

Using eqn. 2.8:

$cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{-5}{\sqrt{13.25}\sqrt{10.25}} = \frac{-5}{11.52} = -0.434$

- The Cross Product: it is product of vectors gives a vector:

$$\vec{A} \times \vec{B} = \vec{C} \qquad (2.9)$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin\theta \qquad (2.10)$$

The direction of $\vec{C}$ is perpendicular on the plane of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \vec{C} = (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k \qquad (2.11)$$

- Del Operator: is a vector differential operator denoted by the symbol $\vec{\nabla}$:

$$\vec{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}k\frac{\partial}{\partial z} \qquad (2.12)$$

a. Gradient of Scalar (pressure)

$$\vec{\nabla}p = i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}k\frac{\partial p}{\partial z} \qquad (vector) \qquad (2.13)$$

b. Divergence of a Vector (velocity)

$$\vec{\nabla}.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (scalar)\ (How?) \qquad (2.14)$$

c. Curl of a Vector (velocity)

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k \qquad (vector) \qquad (2.15)$$

- Laplacian Operator

If Q is any quantity then,

$$\vec{\nabla}.\vec{\nabla}Q \equiv \vec{\nabla}^2 Q \qquad (2.16)$$

where $\vec{\nabla}^2$ (del squared) is the scalar differential operator:

$$\vec{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (2.17)$$

$\vec{\nabla}^2 Q$ is called the Laplacian of Q and appears in several important partial equations of mathematical physics.

Exercises

Q1. What are the possible works of the atmospheric science researchers to get and handle measurements?

Q2. List the main types of models, then explain what are they developed for?

Q3. Define: Numerical model

Q4. What is the deference between the physical error and mathematical error? Give examples.

Q5. Give the expressions of dot product, cross product, and Laplacian operator.

MATLAB Work

Write Matlab code to do the calculations in Example 1.

Homework

1. Prove in detail answering (How?) questions in the lecture.
2. Write Equation 2.1 in another form (in its components form)