Lecture 5

Three Additional Equations

5.1 Closing the System of Equations

So far, we have taken three equations of seven of the closed system of equations that govern the atmospheric dynamics. These three equations are the momentum equations in x-, y-, and z-direction. The four other equations are:

1. The gas equation (or equation of state)
2. The thermodynamic equation
3. The continuity equation
4. The conservation law of water vapor substance.

In this lecture, we will take the first two equations.

5.2 The Gas Equation

The gas equation is:

$$P = \rho \, R \, T$$

where P is pressure ; ρ is density ; $R = 287\, J\, kg^{-1} K^{-1}$; T is temperature

A perfect gas (ideal gas) obeys the physical laws of Boyle and Charles.

Boyle's law:

$$P_1 V_1 = P_2 V_2 \qquad at\ constant\ \ T$$

For a system, if the volume increases, the pressure will decreases, at constant temperature.

Charles' law:

$$\frac{P_1}{T_1} = \frac{P_2}{T_2} \qquad \text{at constant } \ V$$

For a system, if the pressure increases, the temperature will decreases, at constant volume.

where V is volume. Combining the two laws we get:

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} = C$$

Where C is constant depends on the mass of gas and equal to $287\, J\, kg^{-1} K^{-1}$ (specific gas constant).

$$\frac{P\alpha}{T} = R \qquad where \quad \alpha = \frac{1}{\rho} \quad \text{is the specific volume}$$

Thus, $P = \rho RT$

Note: *the specific volume of a substance is the ratio of the substance's volume to its mass and equal to the reciprocal of the density.*

Question: what is the difference between ideal gas and the real gas? *Google it!*

5.3 The Thermodynamic Equation

The thermodynamic energy equation comes from the first law of thermodynamics (conservation of energy):

$$dH = du + dw$$

where: dH is the amount of heat added to system per unit mass.

du is the change in energy per unit mass; $du = C_v\, dT$

dw is the work done by unit mass on a system ; $dw = p\, d\alpha$

$$dH = C_v\, dT + p\, d\alpha \qquad (5.1)$$

Differentiation of the equation of state ($P\alpha = RT$) gives:

$$P\, d\alpha + \alpha\, dP = R\, dT$$

$$P\, d\alpha = R\, dT - \alpha\, dP \qquad (5.2)$$

Substitute (5.2) in (5.1) we get:

$$dH = C_v\, dT + R\, dT - \alpha\, dP$$

$$dH = (C_v + R)\, dT - \alpha\, dP$$

Recall that $R = C_p - C_v \quad \Rightarrow \quad C_p = C_v + R$

where $C_p = 1004\, Jkg^{-1}K^{-1}$ and $C_v = 717\, Jkg^{-1}K^{-1}$ are the specific heat at constants pressure and volume, respectively.

Hence,

$$dH = C_p\, dT - \alpha\, dP \qquad (5.3)$$

Both of equations (5.1) and (5.3) represent the first law of thermodynamics.

5.3.1 Adiabatic Assumption

It is assumed that *dH=0* for most air parcel movements. This assumption can be made whenever the motion is fast so that the heat exchange between the parcel and the surroundings is negligible. (*Why we consider the heat exchange here is negligible?)*

For adiabatic motion, equations of first law of thermodynamics become:

$$C_v\, dT + P\, d\alpha = 0 \qquad (1)$$

$$C_p\, dT - \alpha\, dP = 0 \qquad (2)$$

By substituting ($P = \frac{1}{\alpha} R\, T$) from the equation of state, in equation (1)

$$C_v\, dT + \frac{1}{\alpha} R\, T\, d\alpha = 0$$

$$C_v\, dT = -R\, T \frac{d\alpha}{\alpha}$$

$$C_v\, \frac{dT}{T} = -R\, \frac{d\alpha}{\alpha}$$

$$\int_{T_1}^{T} \frac{dT}{T} = -\frac{R}{C_v} \int_{\alpha_1}^{\alpha} \frac{d\alpha}{\alpha}$$

$$\ln\, (T - T_1) = -\frac{R}{C_v} (\ln(\alpha - \alpha_1))$$

Now, by taking the exponential (e) for the two sides:

$$\frac{T}{T_1} = (\frac{\alpha}{\alpha_1})^{-\frac{R}{C_v}} \qquad (4)$$

If T increases, α will decrease and vice versa. (*Explain*)

Now from equation of state:

$$\alpha = \frac{R\, T}{P} \qquad (5)$$

Substitute (5) in (2) we get:

$$C_p\, dT - \frac{R\,T}{P} dP = 0$$

$$\int_{T_1}^{T} \frac{dT}{T} = \frac{R}{C_p} \int_{P_1}^{P} \frac{dP}{P}$$

$$\ln \frac{T}{T_1} = \frac{R}{C_P} \ln \frac{P}{P_1}$$

and by taking the exponential for both sides:

$$\frac{T}{T_1} = (\frac{P}{P_1})^{\frac{R}{C_P}} \qquad (6)$$

If T increases P will increases and vice versa.

From equation (4) and equation (6) we get:

$$(\frac{\alpha}{\alpha_1})^{-\frac{R}{C_v}} = (\frac{P}{P_1})^{\frac{R}{C_P}}$$

$$\frac{\alpha}{\alpha_1} = (\frac{P}{P_1})^{-\frac{C_v}{C_P}}$$

An increase in P corresponds to a decrease in α and vice versa

Questions:

1. **Explain why the temperature of a gas drops in an adiabatic expansion.**
   Ans. In an adiabatic expansion since no heat is supplied from outside, therefore the energy required for the expansion of the gas is taken from the gas itself. This means that, the internal energy of an ideal gas decreases, and because the internal energy of an ideal gas depends only on the temperature, therefore its temperature must decreases.
2. **Write down the thermodynamic equations (two formulas)**
3. **Derive the thermodynamic equations.**
4. **What is the adiabatic assumption and show how T, α, and P relate with each other.**