1.3. Tautology /Contradiction / Contingency

Definition 1.3.1. (Tautology)

A tautology (theorem or lemma) is a logical proposition that is always true.

Remark 1.3.2. One informal way to check whether or not a certain logical formula is a theorem is to construct its truth table.

Example 1.3.3. $p \lor \sim p$.

Definition 1.3.4. (Contradiction)

A contradiction is a logical proposition that is always false.

Example 1.3.5. $p \land \sim p$.

Definition 1.3.6. (Contingency)

A contingency is a logical proposition that is neither a tautology nor a contradiction.

Example 1.3.7.

(i) The logical proposition $p \lor q \to \sim r$ is a contingency. See Example 1.2.3(i).

(ii) The logical proposition $p \lor \sim (p \land q)$ is a tautology.

p	q	$p \land q$	$\sim (p \land q)$	$p \lor \sim (p \land q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercise 1. 1.3.8

(i) Build a truth table to verify that the logical proposition

$$(p \leftrightarrow q) \wedge (\sim p \wedge q)$$

is a contradiction.

(ii) (Low of Syllogism) Show that the logical proposition

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow q)$$

is a tautology.

Definition 1.3.8. (Logically equivalent)

Propositions r and s are logically equivalent if the truth tables of r and s are the same and denoted by $(r \equiv s)$.

Example 1.3.9. Show that

$$(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q.$$

Solution. Show the truth values of both propositions are identical.

p	q	~q	p → q	~(p → q)	p ∧ ~q
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Theorem 1.3.10. (Relation Between Logical Equivalent and Tautology)

$r \equiv s$ if and only if the statement $r \leftrightarrow s$ is a tautology.

1.3.11. Algebra of Logical Proposition

The logical equivalences below are important equivalences that should be memorized.

1-Identity Laws: $p \wedge T \equiv p$.
$p \vee F \equiv p$.

2-Domination Laws: $p \vee T \equiv T$.
$p \wedge F \equiv F$.

3-Idempotent Laws: $p \vee p \equiv p$.
$p \wedge p \equiv p$.

4- Double Negation Law: $\sim(\sim p) \equiv p$.

5- Commutative Laws: $p \vee q \equiv q \vee p$.
$p \wedge q \equiv q \wedge p$.

6- Associative Laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

7- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

8- De Morgan's Laws: $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

9- Absorption Laws: $p \wedge (p \vee q) \equiv p$.
$p \vee (p \wedge q) \equiv p$.
$p \wedge (\sim p \vee q) \equiv p \wedge q$.

$p \vee (\sim p \wedge q) \equiv p \vee q$.

10-Implication Law: $(p \rightarrow q) \equiv (\sim p \vee q)$.

11- Contrapositive Law: $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.

12- Tautology: $p \vee \sim p \equiv T$.

13- Contradiction: $p \wedge \sim p \equiv F$.

14- Equivalence: $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$.

15- $p \underline{\vee} q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

Solution.

(8) We using truth table to prove $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

p	q	~ p	~q	p ∧ q	~ (p ∧ q)	~ p ∨ ~q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(14) We using truth table to prove $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$.

p	q	p → q	q → p	p → q ∧ q → p	p ↔ q
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F

F	F	T	T	T	T

(15) $p \veebar q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$\mathbf{p \veebar q}$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F