جبر خطي

مرحلة ٤/علوم رياضيات

Linear Algebra with Applications I

References

1) Introductactary Linear Algebra with Applications, by Bernard Kolman.
2) Elementary Linear Algebra, by Howard Anton.
3) Linear Algebra, by Fraleigh
4) Theory and Problems of Linear Algebra, by Seymour Lipschutz.
5) 3000 Solved Problems in Linear Algebra, by Seymour Lipschutz.
6) Introduction to Linear Algebra, by Franz Hohn.

§ 1 : Fields

Definition: A binary operation $*$ on a set X is a function $* : X \times X \longrightarrow X$.

Definition: A set F with two binary operations $+$ and $.$ defined on F is said to be a field if the following properties are satisfied:

1) $a+b \in F \quad \forall a, b \in F$,
2) $a+b = b+a \quad \forall a, b \in F$,
3) $a+(b+c) = (a+b)+c \quad \forall a, b, c \in F$,
4) There exists an element $z \in F$ such that $a+z = z+a \quad \forall a \in F$,
5) $\forall a \in F$ there is an element denoted by $-a \in F$ such that $a+(-a) = (-a)+a = z$,
6) $a.b \in F \quad \forall a, b \in F$,
7) $a \cdot b = b \cdot a \quad \forall a, b \in F$,
8) $a.(b.c) = (a.b).c \quad \forall a, b, c \in F$,
9) There exists an element $u \in F$ such that $u \neq z$ and $a.u = u.a = a \quad \forall a \in F$,
10) $\forall a \in F - \{z\}$ there exists an element denoted by $a^{-1} \in F$ such that $a.a^{-1} = a^{-1}.a = u$,
11) $a.(b+c) = (a.b)+(a.c) \quad \forall a, b, c \in F$.

3. Examples:

1) The set R of all real numbers with the usual addition $+$ and the usual multiplication $.$ of real numbers is a field.

2) The set $Q=\{\frac{p}{q} \mid p$ and q are integers and $q \neq 0\}$ of all rational numbers with the usual addition $+$ and the usual multiplication $.$ of rational numbers is a field.

3) The set $C=\{a+bi \mid a$ and b are real numbers$\}$ of all complex numbers with the usual addition $+$ and the usual multiplication $.$ of the complex numbers is a field (notice that $i.i=-1$ and $(a+bi)^{-1}=(a-bi)/(a^2+b^2)$.

4) The set R of all real numbers with the two binary opertions $*$ and $\circ$ defined by $a*b=a+b+1$ and $a\circ b=a.b+a+b$ $\forall a,b\in F$ is a field.

5) The set $\mathbb{Z}_p=\{0,1,2,\dots,p-1\}$ with the two binary operations $+_p$ (the addition of integers modulo p) and $\cdot_p$ (the multiplication of integers modulo p) is afield where p is any prime number.

6) The set $F_3=\{4,8,12\}$ and $+'_{12}$ and $\cdot'_{12}$ be the two binary operations defined on F_3 as

follows :

$+_{12}$	4	8	12
4	8	12	4
8	12	4	8
12	4	8	12

$\cdot_{12}$	4	8	12
4	4	8	12
8	8	4	12
12	12	12	12

is a field.

§2: Vectors in the Plane

Definition: A vector in the plane is a 2×1 matrix $X=\binom{x}{y}$ where x and y are real numbers. x is called the first component of X and y is called the second compotent.

Remarks:

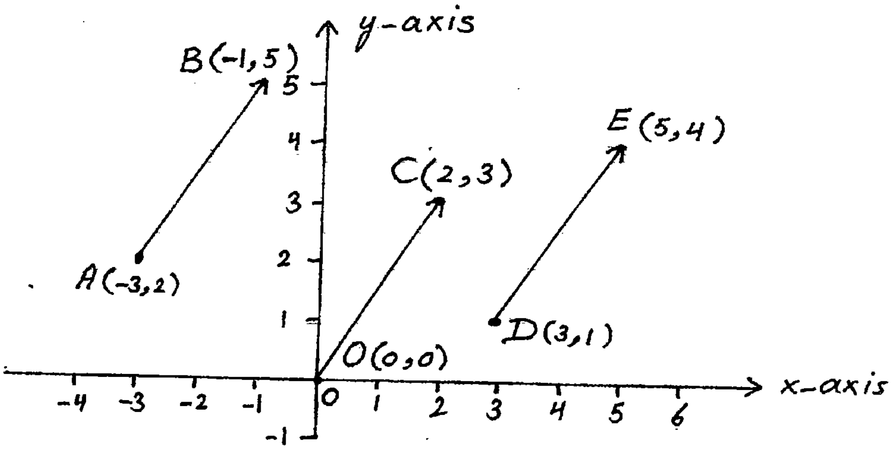
1) We can also write the vector X as $X=(x,y)$.
2) The set of all vectors in the plane (the Euclidean plane) is denoted by R^2.
3) The vector $X=\binom{x}{y}$ (or $X=(x,y)$) is represented by the line segment with initial point $O(0,0)$ and the terminal point $P(x,y)$. The initial point $O(0,0)$ is called the tail of the vector X and the terminal point $P(x,y)$ is called the head

of the vector X, and X is denoted by $\overrightarrow{OP}$

The vector $X = \binom{x}{y}$ is also represented by any line segment with initial point $A = (a_1, a_2)$ and terminal point $B(a_1 + x, a_2 + y)$ and A is the tail of the vector X and B is the head of the vector X, and X is then will be denoted by $\overrightarrow{AB}$.

4) Any vector $X = \binom{x}{y}$ (or $X = (x, y)$) has length or magnitude denoted by $\|X\|$ which is equal $\sqrt{x^2+y^2}$ (i.e. $\|X\| = \sqrt{x^2+y^2}$).

Example: The vector $X = \binom{2}{3}$ can be represented by any of the following three vectors shown in the following figure



Example: Find the length of the vector $X = (7, -3)$.

Solution: The length of the vector X is

6. $\|X\| = \sqrt{7^2+(-3)^2} = \sqrt{49+9} = \sqrt{58}$.

Example: Find the length of the vector $\overrightarrow{PQ}$ where P is the point $(4,3)$ and Q is the point $(2,-9)$.

Solution: $\overrightarrow{PQ} = (2-4, -9-3) = (-2,-12)$.

$\therefore \|\overrightarrow{PQ}\| = \sqrt{(-2)^2+(-12)^2} = \sqrt{4+144} = \sqrt{148}$.

Vector Operations in R^2

Definition: Let $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ be any two vectors in the Euclidean plane. The sum of the vectors X and Y is the vector (x_1+x_2, y_1+y_2) and is denoted by $X+Y$.

Example: Find $X+Y$ where X and Y are the vectors $X=(7,8)$ and $Y=(3,-2)$.

Solution: $X+Y=(7+3, 8-2)=(10,6)$.

Definition: If $X=(x,y)$ be any vector in R^2 and c is any scalar ($c \in R$), then the scalar multiple cX of X by c is the vector (cx, cy) (i.e. $cX=(cx,cy)$).

Example: Find $7X$ where $X=(2,3)$.

7. <u>Solution</u>: $7X = 7(2,3) = (14, 21)$.

<u>Remarks</u>:

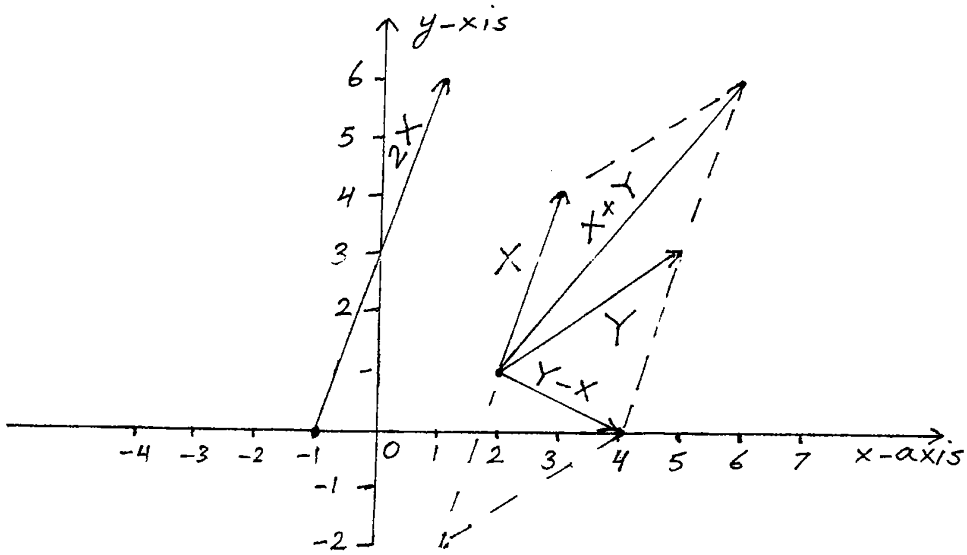
1) The vector $O(0,0)$ is called the zero vector which satisfies that $X + O = X \quad \forall X \in R^2$.
2) $\forall X = (x,y) \in R^2$, we have $(-1)X = -1(x,y) = (-x,-y)$. We write $(-1)X$ as $-X$.
3) $\forall X, Y \in R^2$ we write $X - Y$ instead of $X + (-Y)$.
4) $\forall X \in R^2$, we have $X - X = O$.

<u>Example</u>: Let $X = (1,3)$ and $Y = (3,2)$ and $c = 2$. Find cX, $X+Y$, $Y-X$ and show the answer in a figure.

<u>Solution</u>: $cX = 2X = 2(1,3) = (2,6)$.

$X+Y = (1,3) + (3,2) = (4,5)$.

$Y-X = (3,2) - (1,3) = (2,-1)$.



8. **Definition**: Two vectors $X=(x_1, y_1)$ and $Y=(x_2, y_2)$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$.

Definition: The inner product (or the dot product) of the vectors $X=(x_1, y_1)$ and $Y=(x_2, y_2)$ is $x_1x_2 + y_1y_2$ and is denoted by $X \cdot Y$.

Remarks:

1) $X \cdot Y = \|X\| \, \|Y\| \cos\theta$ where θ is the angle between the nonzero vectors X and Y and $0 \leq \theta \leq 2\pi$.
2) $X \cdot Y = Y \cdot X \quad \forall X, Y \in R^2$.
3) $X \cdot X = \|X\|^2 \geq 0$.
4) $X \cdot X = 0$ iff $X = 0$.
5) $(X+Y) \cdot Z = X \cdot Z + Y \cdot Z$.
6) $(cX) \cdot Y = X \cdot (cY) = c(X \cdot Y)$.

Example: Let $X=(7,4)$ and $Y=(3,9)$. Find $\|X\|$, $\|Y\|$, $X \cdot Y$, and the angle between X and Y.

Solution: $\|X\| = \sqrt{7^2+4^2} = \sqrt{49+16} = \sqrt{65}$.

$\|Y\| = \sqrt{3^2+9^2} = \sqrt{9+81} = \sqrt{90}$.

$X \cdot Y = (7,4) \cdot (3,9) = 21 + 36 = 57$

$$\cos\theta = \frac{X \cdot Y}{\|X\| \, \|Y\|} = \frac{57}{\sqrt{65}\,\sqrt{90}} = \frac{57}{\sqrt{5850}}$$

9. $\therefore\ \theta = 41.82^\circ$ is the angle between X and Y.

Definition: A unit vector is a vector whose length is 1. If X is any nonzero vector, then the vector $U_X = \frac{X}{\|X\|}$ is the unit vector in the direction of X.

Example: Let $X = (8, -6)$. Find the unit vector in the direction of X.

Solution: $\|X\| = \sqrt{8^2 + (-6)^2} = \sqrt{64+36} = \sqrt{100} = 10$.

Thus $U_X = \frac{1}{10}(8, -6) = (0.8, -0.6)$.

Remarks:

1) The unit vector along the positive x-axis is $(1, 0)$ and is denoted by i (i.e. $i = (1, 0)$).
2) The unit vector along the positive y-axis is $(0, 1)$ and is denoted by j (i.e. $j = (0, 1)$).
3) The two nonzero vectors X and Y are perpendicular (or orthogonal) iff $X \cdot Y = 0$.
4) i and j are orthogonal vectors since $i \cdot j = (1, 0) \cdot (0, 1) = 0$
5) We can write any vector $X = (x, y)$ as $X = xi + yj$ since $X = (x, y) = (x, 0) + (0, y)$.

$= x(1,0) + y(0,1) = xi + yj$.

6) i and j are called the standard unit vectors in R^2.

Example: Write the vector $X = (7,12)$ in terms of i and j .

Solution: $X = (7,12) = 7i + 12j$.

Remarks:

1) If the set F with the two binary operations $+$ and $\cdot$ is a field, then this field will be denoted by $(F, +, \cdot)$

2) If $(F, +, \cdot)$ is a field then the set of all vectors $X = (x, y)$ where $x, y \in F$ is denoted by F^2.

Example: For the field $(\mathbb{Z}_7, +_7, \cdot_7)$, let $X = (3,2)$ and $Y = (5,6)$ be two vectors in $\mathbb{Z}_7^2$. Find $3X + 4Y$.

Solution:

$$
\begin{aligned}
3X + 4Y &= 3(3,2) + 4(5,6) \\
&= (3 \cdot_7 3, 3 \cdot_7 2) + (4 \cdot_7 5, 4 \cdot_7 6) \\
&= (2,6) + (6,3) \\
&= (2 +_7 6, 6 +_7 3) \\
&= (1,2)
\end{aligned}
$$

Caution: The definition of the inner product (or the dot product) given in page 8 is not applicable for F^2 if F is neither R nor Q .

§3: Vector Spaces

Definition: Let $(F,+,\cdot)$ be a field.
A vector space V over F is a set V together with a binary operation $\oplus$ and an operation of scalar multiplication of the elements of V by the elements of F, such that the following conditions are satisfied:

1) $x \oplus y \in V \quad \forall x, y \in V$.
2) $x \oplus y = y \oplus x \quad \forall x, y \in V$.
3) $x \oplus (y \oplus w) = (x \oplus y) \oplus w \quad \forall x, y, w \in V$.
4) There is an element $0 \in V$ such that $v \oplus 0 = 0 \oplus v = v \quad \forall v \in V$.
5) $\forall v \in V$ there is an element $-v \in V$ such that $v \oplus -v = -v \oplus v = 0$.
6) $cv \in V \quad \forall c \in F$ and $\forall v \in V$.
7) $c(bv) = (c.b)v \quad \forall c, b \in V$.
8) $(c+b)v = cv \oplus bv \quad \forall c, b \in F$ and $\forall v \in V$.
9) $c(v \oplus w) = cv \oplus cw$
10) $uv = v \quad \forall v \in V$ and u is the multiplication identity of F.

Definitions:
If V is a vector space over a field $(F,+,.)$ then any element in V is called a vector, the binary operation $\oplus$ defined on V is

12. called vector addition, the elements of F are called scalars, the vector 0 is called the zero vector, the vector $-v$ is called the negative vector.

Remark: Every field is a vector space over itself.

Example 1: The set R (of all real numbers) together with the usual addition of real numbers and the usual multiplication of real numbers is a vector space over the usual field of real numbers $(R,+,\cdot)$.

Example 2: The set C (of all complex numbers) together with the usual addition of complex numbers and the usual multiplication of complex numbers is a vector space over the usual field of complex numbers $(C,+,\cdot)$.

Example 3: The set Q (of all rational numbers) together with the usual addition of rational numbers and the usual multiplication of rational numbers is a vector space over the usual field of rational numbers $(Q,+,\cdot)$.

Example 4: The set $\mathbb{Z}_7=\{0,1,2,3,4,5,6\}$

13. together with $+_7$ (the addition of integers modulo 7) and $\cdot_7$ (the multiplication of integers modulo 7) is a vector space over the field $(\mathbb{Z}_7, +_7, \cdot_7)$.

Remark: Let $(F,+,\cdot)$ be a field. The set $F^n=\{(x_1,x_2,\dots,x_n) \mid x_1,x_2,\dots,x_n\in F\}$ together with the usual addition $+' : F^n\times F^n \longrightarrow F^n$ defined by

$(x_1,x_2,\dots,x_n)+(y_1,y_2,\dots,y_n)$
$=(x_1+y_1, x_2+y_2,\dots,x_n+y_n)$, $\forall (x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\in F^n$

and the usual scalar multiplication $\cdot' : F\times F^n \longrightarrow F^n$ defined by $k\cdot'(x_1,x_2,\dots,x_n)=(k\cdot x_1, k\cdot x_2,\dots,k\cdot x_n)$ $\forall k\in F$ and $\forall (x_1,x_2,\dots,x_n)\in F^n$ is a vector space over the field $(F,+,\cdot)$.

Example 5: Show that $\mathbb{Z}_{13}^3$ together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the field $(\mathbb{Z}_{13}, +_{13}, \cdot_{13})$.

Solution:

$\forall\, v=(v_1,v_2,v_3),\ w=(w_1,w_2,w_3),\ y=(y_1,y_2,y_3)\in \mathbb{Z}_{13}^3$ and $\forall\, r,s\in \mathbb{Z}_{13}$, we have:

1) $v+w=(v_1,v_2,v_3)+(w_1,w_2,w_3)$
$\quad =(v_1+_{13}w_1, v_2+_{13}w_2, v_3+_{13}w_3)\in \mathbb{Z}_{13}^3$ since $v_1+_{13}w_1,\ v_2+_{13}w_2,\ v_3+_{13}w_3\in \mathbb{Z}_{13}$,

2) $v+w=(v_1,v_2,v_3)+(w_1,w_2,w_3)$

$$= (v_1 +_{13} w_1 , v_2 +_{13} w_2 , v_3 +_{13} w_3)$$
$$= (w_1 +_{13} v_1 , w_2 +_{13} v_2 , w_3 +_{13} v_3)$$
$$= (w_1 , w_2 , w_3) + (v_1 , v_2 , v_3) = w + v ,$$

3) $(v+w)+y$
$$= ((v_1 , v_2 , v_3) + (w_1 , w_2 , w_3)) + (y_1 , y_2 , y_3)$$
$$= (v_1 +_{13} w_1 , v_2 +_{13} w_2 , v_3 +_{13} w_3) + (y_1 , y_2 , y_3)$$
$$= ((v_1 +_{13} w_1) +_{13} y_1 , (v_2 +_{13} w_2) +_{13} y_2 , (v_3 +_{13} w_3) +_{13} y_3)$$
$$= (v_1 +_{13} (w_1 +_{13} y_1), v_2 +_{13} (w_2 +_{13} y_2), v_3 +_{13} (w_3 +_{13} y_3))$$
$$= (v_1 , v_2 , v_3) + (w_1 +_{13} y_1 , w_2 +_{13} y_2 , w_3 +_{13} y_3)$$
$$= (v_1 , v_2 , v_3) + ((w_1 , w_2 , w_3) + (y_1 , y_2 , y_3))$$
$$= v + (w + y) ,$$

4) $0 = (0,0,0)$ since
$$v + 0 = (v_1 , v_2 , v_3) + (0,0,0)$$
$$= (v_1 +_{13} 0 , v_2 +_{13} 0 , v_3 +_{13} 0)$$
$$= (v_1 , v_2 , v_3) = v ,$$

5) $-v = (-v_1 , -v_2 , -v_3)$ (where $-v_i = 13 - v_i$ when $v_i \neq 0$ while $-0 = 0$) since
$$v + (-v) = (v_1 , v_2 , v_3) + (-v_1 , -v_2 , -v_3)$$
$$= (v_1 +_{13} -v_1 , v_2 +_{13} -v_2 , v_3 +_{13} -v_3)$$
$$= (0,0,0) ,$$

6) $r \cdot v = r \cdot (v_1 , v_2 , v_3)$
$$= (r \cdot_{13} v_1 , r \cdot_{13} v_2 , r \cdot_{13} v_3) \in \mathbb{Z}_{13}^3$$
since $r \cdot_{13} v_1 , r \cdot_{13} v_2 , r \cdot v_3 \in \mathbb{Z}_{13}$,

7) $r \cdot (s \cdot v) = r \cdot (s \cdot (v_1 , v_2 , v_3))$
$$= r \cdot (s \cdot_{13} v_1 , s \cdot_{13} v_2 , s \cdot_{13} v_3)$$
$$= (r \cdot_{13} (s \cdot_{13} v_1), r \cdot_{13} (s \cdot_{13} v_2), r \cdot_{13} (s \cdot_{13} v_3))$$

$= ((r \cdot_{13} s) \cdot_{13} u_1 , (r \cdot_{13} s) \cdot_{13} u_2 , (r \cdot_{13} s) \cdot_{13} u_3)$

$= (r \cdot_{13} s) \cdot (u_1 , u_2 , u_3)$

$= (r \cdot_{13} s) \cdot u$,

8) $(r +_{13} s) \cdot u$

$= (r +_{13} s) \cdot (u_1 , u_2 , u_3)$

$= ((r +_{13} s) \cdot_{13} u_1 , (r +_{13} s) \cdot_{13} u_2 , (r +_{13} s) \cdot_{13} u_3)$

$= ((r \cdot_{13} u_1) +_{13} (s \cdot_{13} u_1) , (r \cdot_{13} u_2) +_{13} (s \cdot_{13} u_2) , (r \cdot_{13} u_3) +_{13} (s \cdot_{13} u_3))$

$= (r \cdot_{13} u_1 , r \cdot_{13} u_2 , r \cdot_{13} u_3) + (s \cdot_{13} u_1 , s \cdot_{13} u_2 , s \cdot_{13} u_3)$

$= r \cdot (u_1 , u_2 , u_3) + s \cdot (u_1 , u_2 , u_3)$

$= r \cdot u + s \cdot u$,

9) $r \cdot (u + w)$

$= r \cdot ((u_1 , u_2 , u_3) + (w_1 , w_2 , w_3))$

$= r \cdot (u_1 +_{13} w_1 , u_2 +_{13} w_2 , u_3 +_{13} w_3)$

$= (r \cdot_{13} (u_1 +_{13} w_1) , r \cdot_{13} (u_2 +_{13} w_2) , r \cdot_{13} (u_3 +_{13} w_3))$

$= ((r \cdot_{13} u_1) +_{13} (r \cdot_{13} w_1) , (r \cdot_{13} u_2) +_{13} (r \cdot_{13} w_2) , (r \cdot_{13} u_3) +_{13} (r \cdot_{13} w_3))$

$= (r \cdot_{13} u_1 , r \cdot_{13} u_2 , r \cdot_{13} u_3) + (r \cdot_{13} w_1 , r \cdot_{13} w_2 , r \cdot_{13} w_3)$

$= r \cdot u_1 + r \cdot u_2$,

10) $1 \cdot u$

$= 1 \cdot (u_1 , u_2 , u_3)$

$= (1 \cdot_{13} u_1 , 1 \cdot_{13} u_2 , 1 \cdot_{13} u_3)$

$= (u_1 , u_2 , u_3) = u$.

Thus $\mathbb{Z}_{13}^3$ together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the Field $(\mathbb{Z}_{13} , +_{13} , \cdot_{13})$.

16. <u>Example 6</u>: Show that Q^4 together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the usual field of rational numbers $(Q,+,\cdot)$.

<u>Solution</u>:

$\forall\, \mathbf{v}=(v_1,v_2,v_3,v_4)$, $\mathbf{w}=(w_1,w_2,w_3,w_4)$, $\mathbf{y}=(y_1,y_2,y_3,y_4)\in Q^4$ and $\forall\, r,s\in Q$, we have:

1) $\mathbf{v}+\mathbf{w}=(v_1,v_2,v_3,v_4)+(w_1,w_2,w_3,w_4)$
$=(v_1+w_1,\ v_2+w_2,\ v_3+w_3,\ v_4+w_4)\in Q^4$
since $v_1+w_1,\ v_2+w_2,\ v_3+w_3,\ v_4+w_4\in Q$,

2) $\mathbf{v}+\mathbf{w}=(v_1,v_2,v_3,v_4)+(w_1,w_2,w_3,w_4)$
$=(v_1+w_1,\ v_2+w_2,\ v_3+w_3,\ v_4+w_4)$
$=(w_1+v_1,\ w_2+v_2,\ w_3+v_3,\ w_4+v_4)$
$=(w_1,w_2,w_3,w_4)+(v_1,v_2,v_3,v_4)=\mathbf{w}+\mathbf{v}$,

3) $(\mathbf{v}+\mathbf{w})+\mathbf{y}$
$=((v_1,v_2,v_3,v_4)+(w_1,w_2,w_3,w_4))+(y_1,y_2,y_3,y_4)$
$=(v_1+w_1,\ v_2+w_2,\ v_3+w_3,\ v_4+w_4)+(y_1,y_2,y_3,y_4)$
$=((v_1+w_1)+y_1,\ (v_2+w_2)+y_2,\ (v_3+w_3)+y_3,\ (v_4+w_4)+y_4)$
$=(v_1+(w_1+y_1),\ v_2+(w_2+y_2),\ v_3+(w_3+y_3),\ v_4+(w_4+y_4))$
$=(v_1,v_2,v_3,v_4)+(w_1+y_1,\ w_2+y_2,\ w_3+y_3,\ w_4+y_4)$
$=(v_1,v_2,v_3,v_4)+((w_1,w_2,w_3,w_4)+(y_1,y_2,y_3,y_4))$
$=\mathbf{v}+(\mathbf{w}+\mathbf{y})$,

4) $\mathbf{0}=(0,0,0,0)$ since
$\mathbf{v}+\mathbf{0}=(v_1,v_2,v_3,v_4)+(0,0,0,0)$

17. 5) $-U = (-u_1, -u_2, -u_3, -u_4)$ since

$U + (-U) = (u_1, u_2, u_3, u_4) + (-u_1, -u_2, -u_3, -u_4)$

$= (u_1 - u_1, u_2 - u_2, u_3 - u_3, u_4 - u_4)$

$= (0, 0, 0, 0) = O$,

6) $r \bullet U = r \bullet (u_1, u_2, u_3, u_4)$

$= (r \cdot u_1, r \cdot u_2, r \cdot u_3, r \cdot u_4) \in \mathbb{Q}^4$ since

$r \cdot u_1, r \cdot u_2, r \cdot u_3, r \cdot u_4 \in \mathbb{Q}$,

7) $r \bullet (s \bullet U) = r \bullet (s \bullet (u_1, u_2, u_3, u_4))$

$= r \bullet (s \cdot u_1, s \cdot u_2, s \cdot u_3, s \cdot u_4)$

$= (r \cdot (s \cdot u_1), r \cdot (s \cdot u_2), r \cdot (s \cdot u_3), r \cdot (s \cdot u_4))$

$= ((r \cdot s) \cdot u_1, (r \cdot s) \cdot u_2, (r \cdot s) \cdot u_3, (r \cdot s) \cdot u_4)$

$= (r \cdot s) \bullet (u_1, u_2, u_3, u_4)$

$= (r \cdot s) \bullet U$

8) $(r + s) \bullet U$

$= (r + s) \bullet (u_1, u_2, u_3, u_4)$

$= ((r+s) \cdot u_1, (r+s) \cdot u_2, (r+s) \cdot u_3, (r+s) \cdot u_4)$

$= (r \cdot u_1 + s \cdot u_1, r \cdot u_2 + s \cdot u_2, r \cdot u_3 + s \cdot u_3, r \cdot u_4 + s \cdot u_4)$

$= (r \cdot u_1, r \cdot u_2, r \cdot u_3, r \cdot u_4) + (s \cdot u_1, s \cdot u_2, s \cdot u_3, s \cdot u_4)$

$= r \bullet (u_1, u_2, u_3, u_4) + s \bullet (u_1, u_2, u_3, u_4)$

$= r \bullet U + s \bullet U$

9) $r \bullet (U + W)$

$= r \bullet ((u_1, u_2, u_3, u_4) + (w_1, w_2, w_3, w_4))$

$= r \bullet (u_1 + w_1, u_2 + w_2, u_3 + w_3, u_4 + w_4)$

$= (r \cdot (u_1 + w_1), r \cdot (u_2 + w_2), r \cdot (u_3 + w_3), r \cdot (u_4 + w_4))$

$= (r \cdot u_1 + r \cdot w_1, r \cdot u_2 + r \cdot w_2, r \cdot u_3 + r \cdot w_3, r \cdot u_4 + r \cdot w_4)$

$= (r \cdot u_1, r \cdot u_2, r \cdot u_3, r \cdot u_4) + (r \cdot w_1, r \cdot w_2, r \cdot w_3, r \cdot w_4)$

$= r\cdot(u_1, u_2, u_3, u_4) + r\cdot(w_1, w_2, w_3, w_4)$

$= r\cdot u + r\cdot w$

10) $1\cdot u$

$= 1\cdot(u_1, u_2, u_3, u_4)$

$= (1\cdot u_1, 1\cdot u_2, 1\cdot u_3, 1\cdot u_4)$

$= (u_1, u_2, u_3, u_4)$

$= u$

Thus Q^4 together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the usual field of rational numbers $(Q, +, \cdot)$.

Remark: Let $(F, +, \cdot)$ be a field. The set $V = M_{m\times n}(F)$ of all $m\times n$ matrices over the field $(F, +, \cdot)$ (i.e. the entries of the matrices belong to F) together with the usual addition of matrices and the usual scalar multiplication of matrices by scalars is a vector space over the field $(F, +, \cdot)$.

Example 7: Show that $M_{2\times 2}(R)$ together with the usual addition of matrices and the usual scalar multiplication of matrices by scalars is a vector space over the

19. usual field of real numbers $(R, +, \cdot)$.

Solution:

$\forall\, U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in M_{2\times2}(R)$ and $\forall\, r, s \in R$, we have

1) $U + W = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} u_{11}+w_{11} & u_{12}+w_{12} \\ u_{21}+w_{21} & u_{22}+w_{22} \end{pmatrix}$

$\in M_{2\times2}(R)$ since $u_{11}+w_{11}, u_{12}+w_{12}, u_{21}+w_{21}, u_{22}+w_{22} \in R$,

2)
$$\begin{aligned} U + W &= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \\ &= \begin{pmatrix} u_{11}+w_{11} & u_{12}+w_{12} \\ u_{21}+w_{21} & u_{22}+w_{22} \end{pmatrix} \\ &= \begin{pmatrix} w_{11}+u_{11} & w_{12}+u_{12} \\ w_{21}+u_{21} & w_{22}+u_{22} \end{pmatrix} \\ &= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \\ &= W + U , \end{aligned}$$

3) $(U + W) + Y$

$$\begin{aligned} &= \left(\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \right) + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \\ &= \begin{pmatrix} u_{11}+w_{11} & u_{12}+w_{12} \\ u_{21}+w_{21} & u_{22}+w_{22} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \\ &= \begin{pmatrix} (u_{11}+w_{11})+y_{11} & (u_{12}+w_{12})+y_{12} \\ (u_{21}+w_{21})+y_{21} & (u_{22}+w_{22})+y_{22} \end{pmatrix} \end{aligned}$$

20.

$$= \begin{pmatrix} u_{11}+(w_{11}+y_{11}) & u_{12}+(w_{12}+y_{12}) \\ u_{21}+(w_{21}+y_{21}) & u_{22}+(w_{22}+y_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} w_{11}+y_{11} & w_{12}+y_{12} \\ w_{21}+y_{21} & w_{22}+y_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \left(\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \right)$$

$$= U + (W + Y) \quad ,$$

4) $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ since

$$U + O = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u_{11}+0 & u_{12}+0 \\ u_{21}+0 & u_{22}+0 \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = U \quad ,$$

5) $-U = \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$ since

$$U + (-U) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11}-u_{11} & u_{12}-u_{12} \\ u_{21}-u_{21} & u_{22}-u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O \quad ,$$

6) $r \cdot U = r \cdot \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} r.u_{11} & r.u_{12} \\ r.u_{21} & r.u_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$

21. since $r \cdot U_{11}, r \cdot U_{12}, r \cdot U_{21}, r \cdot U_{22} \in R$,

7) $$r \cdot (s \bullet U) = r \cdot \left(s \bullet \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\right)$$

$$= r \cdot \begin{pmatrix} s \cdot U_{11} & s \cdot U_{12} \\ s \cdot U_{21} & s \cdot U_{22} \end{pmatrix}$$

$$= \begin{pmatrix} r \cdot (s \cdot U_{11}) & r \cdot (s \cdot U_{12}) \\ r \cdot (s \cdot U_{21}) & r \cdot (s \cdot U_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} (r \cdot s) \cdot U_{11} & (r \cdot s) \cdot U_{12} \\ (r \cdot s) \cdot U_{21} & (r \cdot s) \cdot U_{22} \end{pmatrix}$$

$$= (r \cdot s) \bullet \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

$$= (r \cdot s) \bullet U ,$$

8) $(r+s) \bullet U$

$$= (r+s) \bullet \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (r+s) \cdot U_{11} & (r+s) \cdot U_{12} \\ (r+s) \cdot U_{21} & (r+s) \cdot U_{22} \end{pmatrix}$$

$$= \begin{pmatrix} r \cdot U_{11} + s \cdot U_{11} & r \cdot U_{12} + s \cdot U_{12} \\ r \cdot U_{21} + s \cdot U_{21} & r \cdot U_{22} + s \cdot U_{22} \end{pmatrix}$$

$$= \begin{pmatrix} r \cdot U_{11} & r \cdot U_{12} \\ r \cdot U_{21} & r \cdot U_{22} \end{pmatrix} + \begin{pmatrix} s \cdot U_{11} & s \cdot U_{12} \\ s \cdot U_{21} & s \cdot U_{22} \end{pmatrix}$$

$$= r \bullet \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} + s \bullet \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = r \cdot U + r \cdot s ,$$

22. 9) $r\cdot(v+w)$

$$= r\cdot\left(\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix}\right)$$

$$= r\cdot\begin{pmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot(v_{11}+w_{11}) & r\cdot(v_{12}+w_{12}) \\ r\cdot(v_{21}+w_{21}) & r\cdot(v_{22}+w_{22})\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot v_{11}+r\cdot w_{11} & r\cdot v_{12}+r\cdot w_{12} \\ r\cdot v_{21}+r\cdot w_{21} & r\cdot v_{22}+r\cdot w_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot v_{11} & r\cdot v_{12} \\ r\cdot v_{21} & r\cdot v_{22}\end{pmatrix} + \begin{pmatrix} r\cdot w_{11} & r\cdot w_{12} \\ r\cdot w_{21} & r\cdot w_{22}\end{pmatrix}$$

$$= r\cdot\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} + r\cdot\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix}$$

$$= r\cdot v + r\cdot w$$

10) $1\cdot v = 1\cdot\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} = v$

Thus $M_{2\times 2}(R)$ together with the usual addition of matrices and the usual scalar multiplication of matrices by scalar is a vector space over the usual field of real numbers $(R,+,\cdot)$.

Remark: Let $(F,+,\cdot)$ be a field. The set $F[x]$ of all polynomials in x over the field

23. $(F,+,\cdot)$ (i.e. $F[x]$ is the set of all functions that can be expressed as $p(x)=a_n x^n + a_{n-1}x^{n-1}+\dots + a_1 x + a_0$ where $a_0, a_1, \dots, a_n \in F$) is a vector space over the field $(F,+,\cdot)$.

Remarks:

1) Every vector space V over the usual field of complex numbers is also a vector space over the usual field of real numbers and also a vector space over the usual field of rational numbers.

2) Every vector space V over the usual field of real numbers is also a vector space over the usual field of rational numbers.

Theorem 1: Let the set V together with the binary operation $+$ and the scalar multiplication $\bullet$ (multiplication of vectors by scalars) be a vector space over a field $(F,+,\cdot)$, then:

i) The identity element (with respect to $+$) is unique.

ii) Each element v of V has exactly one inverse element (with respect to $+$).

iii) If $v, w, y \in V$ and $y+v=y+w$ then $v=w$.

iv) If $v, w, y \in V$ and $v+y=w+y$ then $v=w$.

v) $z \bullet v = 0 \quad \forall v \in V$.

24. vi) $k \cdot 0 = 0 \quad \forall k \in F$.

vii) If $k \cdot v = 0$ then $k = z$ or $v = 0$.

viii) $(-u) \cdot v = -v \quad \forall v \in V$.

Proof:

i) Assume that V has two identity elements 0 and e, then we have

$$\left.\begin{array}{l} v + 0 = 0 + v = v \\ \text{and } v + e = e + v = v \end{array}\right\} \forall v \in V.$$

Hence $e = 0 + e = 0$.

Thus the identity element is unique

ii) Let $v \in V$ and assume that v has two inverses $-v$ and w, then we have

$v + -v = -v + v = 0$

and $v + w = w + v = 0$

Hence
$$\begin{aligned} -v &= -v + 0 \\ &= -v + (v + w) \\ &= (-v + v) + w \\ &= 0 + w \\ &= w. \end{aligned}$$

Thus v has only one inverse element.

iii) If $y + v = y + w$, then

$$\begin{aligned} v &= 0 + v \\ &= (-y + y) + v \end{aligned}$$

25. $= -y + (y + v)$

$= -y + (y + w)$

$= (-y + y) + w$

$= 0 + w$

$= w$.

iv) If $v + y = w + y$, then

$v = v + 0$

$= v + (y + -y)$

$= (v + y) + -y$

$= (w + y) + -y$

$= w + (y + -y)$

$= w + 0$

$= w$

v) $0 + (z \cdot v)$

$= z \cdot v$

$= (z + z) \cdot v$

$= (z \cdot v) + (z \cdot v) \quad \forall v \in V$

Then by (iv) above $0 = z \cdot v \quad \forall v \in V$.

vi) $0 + (k \cdot 0)$

$= k \cdot 0$

$= k \cdot (0 + 0)$

$= (k \cdot 0) + (k \cdot 0) \quad \forall k \in F$

Then by (iv) above $0 = k \cdot 0 \quad \forall k \in F$

26\. vii) By (v) above $k \cdot v = 0$ when $k = z$

Assume that $k \cdot v = 0$ and $k \neq z$, then

$$0 = \frac{1}{k} \cdot 0 \qquad \text{by (vi) above}$$

$$= \frac{1}{k} \cdot (k \cdot v)$$

$$= (\frac{1}{k} \cdot k) \cdot v$$

$$= u \cdot v$$

$$= v$$

viii) $(-u) \cdot v$

$$= (-u) \cdot v + 0$$

$$= (-u) \cdot v + (v + -v)$$

$$= ((-u) \cdot v + v) + -v$$

$$= ((-u) \cdot v + u \cdot v) + -v$$

$$= (-u + u) \cdot v + -v$$

$$= z \cdot v + -v$$

$$= 0 + -v$$

$$= -v$$

§4: Subspaces

Definition: Let V together with a binary operation $+$ defined on V and a scalar

27. multiplication $\cdot$ be a vector space over a field $(F,+,\cdot)$ and let W be a nonempty subset of V. If W is a vector space over the same field $(F,+,\cdot)$ with respect to the same operations $+$ and $\cdot$ of the vector space V, then W is called a subspace of V.

Theorem 2: W is a subspace of a vector space V over a field $(F,+,\cdot)$ iff

i) W is a nonempty subset of V (i.e. $\phi \neq W \subseteq V$).

ii) W is closed under the binary operation $+$ defined on V (i.e. $v + w \in W \quad \forall v, w \in W$).

iii) W is closed under the scalar multiplication $\cdot$ defined on $F \times V$ (i.e. $k \cdot v \in W \quad \forall k \in F$ and $\forall v \in W$)

Proof: If W is a subspace of V then by the definition of subspace $W \subseteq V$ and W is a vector space under the binary operation $+$ defined on V and the scalar multiplication $\cdot$ defined on $F \times V$ (i.e. $\cdot : F \times V \longrightarrow V$). Hence by the definition of vector space W is nonempty which is a subset of V (i.e. (i) is satisfied), and

28. $v + w \in W$ $\forall v, w \in W$ by the first property of the definition of vector space (i.e. (ii) is satisfied) and $k \cdot v \in W$ $\forall k \in F$ and $\forall v \in W$ by the sixth property of the definition of vector space (i.e. (iii) is satisfied).

Conversely, if W satisfies (i), (ii), and (iii) then $W \neq \phi$ and

1) $\forall v, w$ we have $v + w \in W$ by (i).
2) $\forall v, w \in W$ we have $v, w \in V$ (since $W \subseteq V$) and hence $v + w = w + v$.
3) $\forall v, w, y \in W$ we have $v, w, y \in V$ (since $W \subseteq V$) and hence $v + (w + y) = (v + w) + y$.
4) Since $W \neq \phi$ then there exists $w \in W$. Hence by (iii) $z \cdot w = 0 \in W$ and $\forall v \in W$ we have $v \in V$ (since $W \subseteq V$) and thus $v + 0 = v$.
5) $\forall v \in W$ we have by (iii) $-u \cdot v = -v$ and $v + -v = 0$.
6) $\forall k \in F$ and $\forall v \in W$ we have $k \cdot v \in W$ by (iii).
7) $\forall k \in F$ and $\forall v, w \in W$ we have $v, w \in V$, hence $k \cdot (v + w) = k \cdot v + k \cdot w$.
8) $\forall k, l \in F$ and $\forall v \in W$ we have $v \in V$, hence $(k + l) \cdot v = k \cdot v + l \cdot v$.
9) $\forall k, l \in F$ and $\forall v \in W$ we have $v \in V$, hence $(k \cdot l) \cdot v = k \cdot (l \cdot v)$.

29. 10) $\forall v \in W$ we have $v \in V$, hence $u \bullet v = v$.

Example 1: List all the subspaces of the vector space R^2 over $(R,+,.)$.

Solution: The subspaces of the vector space R^2 over $(R,+,.)$ are:

1) $\{(0,0)\}$ is a subspace.
2) $\forall k,l \in R \ni (k,l) \neq (0,0)$ we have $W_{k,l} = \{(kt,lt) \mid t \in R\}$ is a subspace. (i.e. every straight line passing through the origin $O(0,0)$ of the plane R^2 is a subspace of R^2 over $(R,+,.)$).
3) R^2 is a subspace.

Example 2: List all the subspaces of the vector space R^3 over $(R,+,.)$.

Solution: The subspaces of the vector space R^3 over $(R,+,.)$ are:

1) $\{(0,0,0)\}$ is a subspace.
2) $\forall k,l,m \in R \ni (k,l,m) \neq (0,0,0)$ we have $W_{k,l,m} = \{(kt,lt,mt) \mid t \in R\}$ is a subspace. (i.e. every straight line passing through the origin $O(0,0,0)$ of the space R^3 over $(R,+,.)$).
3) $\forall k,l,m \in R \ni (k,l,m) \neq (0,0,0)$ we

30. have $V_{k,l,m} = \{(x,y,z) \in R^3 \mid kx + lm + mz = 0\}$ is a subspace. (i.e. every plane passing through the origin $O(0,0,0)$ of the space R^3 is a subspace R^3 over $(R,+,\cdot)$).
4) R^3 is a subspace.

Example 3: Show that $W = \{(x,0,z) \mid x, z \in R\}$ is a subspace of the vector space R^3 over $(R,+,\cdot)$.
Solution: The vector $(0,0,0) \in W$ since the second component is 0. Thus $W \neq \phi$.
$\forall\, v = (v_1, 0, v_3),\ w = (w_1, 0, w_3) \in W$ we have
$v + w = (v_1, 0, v_3) + (w_1, 0, w_3)$
$\quad = (v_1 + w_1, 0, v_3 + w_3) \in W$ since $v_1 + w_1$, $v_3 + w_3 \in R$ and the second component is 0.
$\forall k \in R$ and $\forall\, v = (v_1, 0, v_3) \in W$ we have
$k \cdot v = k \cdot (v_1, 0, v_3) = (kv_1, 0, kv_3) \in W$ since $kv_1, kv_3 \in R$ and the second compotent is 0.
Thus by theorem 2, W is a subspace of R^3 over $(R,+,\cdot)$.

Example 4: Show that $W = \{(x,y,z) \mid x + y + 2z = 0\}$ is a subspace of the vector space R^3 over $(R,+,\cdot)$.
Solution: The vector $(0,0,0) \in W$ since

31. $0+0+2\times 0=0$. Thus $W\neq\phi$.

$\forall\, u=(u_1,u_2,u_3),\ w=(w_1,w_2,w_3)\in W$ we have

$u+w=(u_1,u_2,u_3)+(w_1,w_2,w_3)$

$\quad=(u_1+w_1,\ u_2+w_2,\ u_3+w_3)\in W$ since

$(u_1+w_1)+(u_2+w_2)+2(u_3+w_3)$

$=u_1+w_1+u_2+w_2+2u_3+2w_3$

$=u_1+u_2+2u_3+w_1+w_2+2w_3$

$=0+0$ since $u=(u_1,u_2,u_3),\ w=(w_1,w_2,w_3)$ $\in W$ which means that $u_1+u_2+2u_3=0$, $w_1+w_2+2w_3=0$

$=0$

$\forall\, k\in R$ and $\forall\, u=(u_1,u_2,u_3)\in W$ we have

$k\cdot u=k\cdot(u_1,u_2,u_3)$

$\quad=(ku_1,\ ku_2,\ ku_3)\in W$ since

$ku_1+ku_2+2ku_3$

$=k(u_1+u_2+2u_3)$

$=k\cdot 0$ since $(u_1,u_2,u_3)\in W$ which means that $u_1+u_2+2u_3=0$

$=0$

Thus by theorem 2, W is a subspace of R^3 over $(R,+,\cdot)$.

Example 5: Show that $W=\left\{\begin{pmatrix} t & s \\ 3t & 5t\end{pmatrix}\,\middle|\, t,s\in R\right\}$ is a subspace of the vector space $M_{2\times 2}(R)$ over $(R,+,\cdot)$.

Solution: The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}\in W$. Thus $W\neq\phi$.

32. $\forall \begin{pmatrix} t_1 & s_1 \\ 3t_1 & 5t_1 \end{pmatrix}, \begin{pmatrix} t_2 & s_2 \\ 3t_2 & 5t_2 \end{pmatrix} \in W$ we have

$$\begin{pmatrix} t_1 & s_1 \\ 3t_1 & 5t_1 \end{pmatrix} + \begin{pmatrix} t_2 & s_2 \\ 3t_2 & 5t_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 & s_1 + s_2 \\ 3t_1 + 3t_2 & 5t_1 + 5t_2 \end{pmatrix}$$

$$= \begin{pmatrix} t_1 + t_2 & s_1 + s_2 \\ 3(t_1 + t_2) & 5(t_1 + t_2) \end{pmatrix} \in W \quad \text{since } t_1 + t_2, s_1 + s_2 \in R.$$

$\forall k \in R$ and $\forall \begin{pmatrix} t & s \\ 3t & 5t \end{pmatrix} \in W$ we have

$$k \cdot \begin{pmatrix} t & s \\ 3t & 5t \end{pmatrix} = \begin{pmatrix} k.t & ks \\ k.3t & k \cdot 5t \end{pmatrix}$$

$$= \begin{pmatrix} kt & ks \\ 3kt & 5kt \end{pmatrix} \in W \quad \text{since } kt, ks \in R.$$

§5 : Bases and dimension

Definition: Let V be a vector space over a field $(F, +, \cdot)$. A vector w is called a linear combination of the vectors $v_1, v_2, \ldots, v_m$ if $w = k_1 v_1 + k_2 v_2 + \cdots + k_m v_m$ for some $k_1, k_2, \ldots, k_m \in F$.

Example 1: In the vector space R^3 over $(R, +, \cdot)$ we have $w = (2, 6, 1)$ is a linear combination of $v_1 = (1, 1, 0)$, $v_2 = (2, 0, 3)$, and $v_3 = (1, 3, 4)$ since

$3v_1 - v_2 + v_3 = 3(1, 1, 1) - (2, 0, 3) + (1, 3, 4)$

$= (3,3,0) - (2,0,3) + (1,3,4)$
$= (3-2+1\ ,\ 3-0+3\ ,\ 0-3+4)$
$= (2,6,1)$.

Example 2: In the vector space R^3 over $(R,+,\cdot)$, show that $w = (5,7,-1)$ is a linear combination of the vectors $u_1 = (1,1,1)$, $u_2 = (1,0,1)$, and $u_3 = (2,1,3)$.

Solution: To show that w is a linear combination of u_1, u_2, u_3 we should find $k_1, k_2, k_3 \in R$ such that $k_1u_1 + k_2u_2 + k_3u_3 = w$.

Now $k_1u_1 + k_2u_2 + k_3u_3 = w$

$\Rightarrow k_1(1,1,1) + k_2(1,0,1) + k_3(2,1,3) = (5,7,-1)$

$\Rightarrow (k_1,k_1,k_1) + (k_2,0,k_2) + (2k_3,k_3+3k_3) = (5,7,-1)$

$\Rightarrow (k_1+k_2+2k_3\ ,\ k_1+0+k_3\ ,\ k_1+k_2+3k_3) = (5,7,-1)$

$$\Rightarrow \begin{aligned} k_1 + k_2 + 2k_3 &= 5 \\ k_1 \quad\ + k_3 &= 7 \\ k_1 + k_2 + 3k_3 &= -1 \end{aligned}$$

by using Cramer's rule we have

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1\cdot(-1) - 1\cdot(2) + 2\cdot(1) = -1-2+2 = -1 .$$

$$|A_1| = \begin{vmatrix} 5 & 1 & 2 \\ 7 & 0 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 5 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 7 & 1 \\ -1 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} 7 & 0 \\ -1 & 1 \end{vmatrix}$$

34. $= 5\cdot(-1) - 1\cdot(22) + 2\cdot(7) = -5 - 22 + 14 = -13$

$$|A_2| = \begin{vmatrix} 1 & 5 & 2 \\ 1 & 7 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 1\cdot\begin{vmatrix} 7 & 1 \\ -1 & 3 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix}$$

$$= 1\cdot(22) - 5(2) + 2(-8) = 22 - 10 - 16 = -4$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 0 & 7 \\ 1 & 1 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} 0 & 7 \\ 1 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix} + 5\cdot\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1\cdot(-7) - 1\cdot(-8) + 5\cdot(1) = -7 + 8 + 5 = 6.$$

Thus $k_1 = \dfrac{|A_1|}{|A|} = \dfrac{-13}{-1} = 13$, $k_2 = \dfrac{|A_2|}{|A|} = \dfrac{-4}{-1} = 4$,

and $k_3 = \dfrac{|A_3|}{|A|} = \dfrac{6}{-1} = -6$.

Hence $w = 13u_1 + 4u_2 - 6u_3$.

Definition: Let V be a vector space over a field $(F, +, \cdot)$ and let $S = \{u_1, u_2, \dots, u_m\} \subseteq V$, then the subspace W of V consisting of all linear combinations of the vectors in S is called the subspace (or the vector space) spanned by $u_1, u_2, \dots, u_m$ and we say that the vectors $u_1, u_2, \dots, u_m$ span W, and we denoted it by $W = \text{span}(S)$ or $W = \text{span}\{u_1, u_2, \dots, u_m\}$.

Example: Let $W = \{(x, y, 0) \mid x, y \in R\}$ is a subspace of R^3 over $(R, +, \cdot)$. Show that $W = \text{span}(S)$, where $S = \{(1,0,0), (0,1,0)\}$.

Solution: Let $v = (x_1, y_1, 0)$ be any vector belongs to W then

$v = (x_1, y_1, 0) = x_1 \cdot (1,0,0) + y_1 (0,1,0)$.

Thus every vector belongs to W is a linear combination of the vectors in S.

Therefore $W = \text{span}(S)$.

Definition: If $S = \{v_1, v_2, \ldots, v_m\}$ is a nonempty set of vectors, then the vector equation

$k_1 v_1 + k_2 v_2 + \cdots + k_m v_m = 0$

has at least one solution which is

$k_1 = k_2 = \cdots = k_m = z$.

If this is the only solution, then S is called a linearly independent set. If there are other solutions, then S is called a linearly dependent set.

Theorem 3: Let V be a vector space over a field $(F, +, \cdot)$.

i- A subset S of V with more than one element is linearly dependent iff at least one of the vectors is a linear combination of the other vectors in S.

ii- A subset S of V with more than one element is linearly independent iff no vector in S is a linear combination of the other vectors in S.

iii- A finite subset S of V that contains the zero vector 0 is linearly dependent.

iv- A subset S of V with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other.

Example 1: If $v_1 = (3, 1, 5)$ and $v_2 = (9, 3, 15)$ then the set $S = \{v_1, v_2\}$ in the vector space R^3 over the field $(R, +, \cdot)$ is linearly dependent since $v_2 = 3v_1$.

Example 2: If $v_1 = (2, 3, 4)$, $v_2 = (1, 1, 2)$ and $v_3 = (3, 5, 6)$ then the set $S = \{v_1, v_2, v_3\}$ in the vector space R^3 over the field $(R, +, \cdot)$ is linearly dependent since $2v_1 - v_2 - v_3 = (0, 0, 0) = 0$.

Example 3: Let $Q[x]$ be the set of all polynomials in x with rational coefficients.

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37\. $Q[x]$ is a vector space over the field $(Q, +, \cdot)$

i- Prove or disprove that the set $S = \{p_1(x), p_2(x), p_3(x)\}$ which consists of the polynomials $p_1(x) = 1 - x$, $p_2(x) = 2 + 3x - 2x^2$ and $p_3(x) = -\frac{1}{2} + 3x - x^2$ is linearly dependent.

ii- Prove or disprove that the set $T = \{p_1(x), p_2(x)\}$ where $p_1(x)$ and $p_2(x)$ are the same polynomials given in (i) above is linearly dependent.

Solution:

i- $k_1 p_1(x) + k_2 p_2(x) + k_3 p_3(x) = 0$

$$\Rightarrow k_1(1 - x) + k_2(2 + 3x - 2x^2) + k_3(-\tfrac{1}{2} + 3x - x^2) = 0$$

$$\Rightarrow k_1 - k_1 x + 2k_2 + 3k_2 x - 2k_2 x^2 - \tfrac{1}{2}k_3 + 3k_3 x - k_3 x^2 = 0$$

$$\Rightarrow (k_1 + 2k_2 - \tfrac{1}{2}k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

$$\Rightarrow \begin{aligned} k_1 + 2k_2 - \tfrac{1}{2}k_3 &= 0 \quad \text{---} (1) \\ -k_1 + 3k_2 + 3k_3 &= 0 \quad \text{---} (2) \\ -2k_2 - k_3 &= 0 \quad \text{---} (3) \end{aligned}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & -\frac{1}{2} \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \cdot (3) - 2 \cdot (1) - \tfrac{1}{2}(2) = 3 - 2 - 1 = 0$$

which means that we should take only two of the equations (1), (2), and (3) to find the values of k_1, k_2, and k_3, we will take

38. ① and ②

$\Rightarrow \quad k_1 + 2k_2 = \frac{1}{2}k_3$

$\quad -k_1 + 3k_2 = -3k_3$

$\Rightarrow |B| = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3+2 = 5$

$|B_1| = \begin{vmatrix} \frac{1}{2}k_3 & 2 \\ -3k_3 & 3 \end{vmatrix} = 7.5\, k_3$

$|B_2| = \begin{vmatrix} 1 & \frac{1}{2}k_3 \\ -1 & -3k_3 \end{vmatrix} = -2.5\, k_3$

$\therefore \quad k_1 = \dfrac{|B_1|}{|B|} = \dfrac{7.5\, k_3}{5} = 1.5\, k_3$

$k_2 = \dfrac{|B_2|}{|B|} = \dfrac{-2.5k_3}{5} = -0.5\, k_3$

take $k_3 = 2$ then $k_1 = 3$ and $k_2 = -1$.
Thus $3p_1(x) - p_2(x) + 2p_3(x) = 0$ which means that $p_1(x), p_2(x), p_3(x)$ are linearly dependent, i.e. S is linearly dependent

ii- $k_1 p_1(x) + k_2 p(x) = 0$

$\Rightarrow k_1(1-x) + k_2(2+3x-2x^2) = 0$

$\Rightarrow k_1 - k_1x + 2k_2 + 3k_2x - 2k_2x^2 = 0$

$\Rightarrow (k_1+2k_2) + (-k_1+3k_2)x - 2k_2x^2 = 0$

$\Rightarrow \quad k_1 + 2k_2 = 0 \quad \ldots ④$

$\quad -k_1 + 3k_2 = 0 \quad \ldots ⑤$

$\quad -2k_2 = 0 \quad \ldots ⑥$

multiply equation ⑥ by $-\frac{1}{2}$, we get

$k_2 = 0 \quad \cdots (7)$

substitute (7) in (4), we get

$k_1 = 0$

Thus $k_1 = 0$ and $k_2 = 0$ which means that $p_1(x)$ and $p_2(x)$ are linearly independent, i.e. T is linearly independent.

Definition: Let V be a vector space over a field $(F, +, \cdot)$ and $S = \{v_1, v_2, \cdots, v_n\}$ is a set of vectors in V, then S is called a basis for V if S is linearly independent and S spans V.

Theorem 4: If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for a vector space V over a field $(F, +, \cdot)$, then every vector v in V can be pressed in the form $v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$ in exactly one way.

Proof: Every vector v in V can be written as a linear combinations of the vectors in S since S spans V.

Now assume that there is a vector w in V and w can be expressed as a linear combination in two ways as follows

$w = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n \quad \cdots (1)$ and

$w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \cdots (2)$

subtracting equation ② from equation ①, we get

$(k_1 - c_1)u_1 + (k_2 - c_2)u_2 + \cdots + (k_n - c_n)u_n = 0$

the $k_1 - c_1 = 0$, $k_2 - c_2 = 0$, $\ldots$, $k_n - c_n = 0$ since S is linearly independent.

Thus $k_1 = c_1$, $k_2 = c_2$, $\ldots$, $k_n = c_n$. Hence w can be expressed in exactly one way as a linear combination of the vectors in S.

Example 1: Prove that $S = \{i, j, k\}$ is a basis for the vector space R^3 over the field $(R, +, \cdot)$, where $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$.

Solution: S is linearly independent since

$c_1 i + c_2 j + c_3 k = (0, 0, 0)$

$\Rightarrow c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$

$\Rightarrow (c_1, 0, 0) + (0, c_2, 0) + (0, 0, c_3) = (0, 0, 0)$

$\Rightarrow (c_1, c_2, c_3) = (0, 0, 0)$

$\Rightarrow c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

Also S spans R^3 since $\forall u = (u_1, u_2, u_3) \in R^3$ we have

$$\begin{aligned} u &= (u_1, u_2, u_3) \\ &= (u_1, 0, 0) + (0, u_2, 0) + (0, 0, u_3) \\ &= u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) \\ &= u_1 i + u_2 j + u_3 k \end{aligned}$$

Thus S is a basis for the vector space R^3

over the field $(R,+,\cdot)$.

Definition: The set $S=\{e_1,e_2,\dots,e_n\}$ where $e_1=(1,0,0,\dots,0)$, $e_2=(0,1,0,\dots,0)$, $\dots$, $e_n=(0,0,\dots,0,1)$ ($e_1,e_2,\dots,e_n$ are n-tuple) is a basis of R^n over the field $(R,+,\cdot)$ which is called the standard basis of R^n over the field $(R,+,\cdot)$.

Example 2: Let $u_1=(1,1,0)$, $u_2=(0,1,1)$, $u_3=(1,0,1)$. Prove that $S=\{u_1,u_2,u_3\}$ is a basis for the vector space R^3 over the field $(R,+,\cdot)$.

Solution:

$k_1u_1+k_2u_2+k_3u_3=(0,0,0)$.

$\Rightarrow k_1(1,1,0)+k_2(0,1,1)+k_3(1,0,1)=(0,0,0)$

$\Rightarrow (k_1,k_1,0)+(0,k_2,k_2)+(k_3,0,k_3)=(0,0,0)$

$\Rightarrow (k_1+k_3,k_1+k_2,k_2+k_3)=(0,0,0)$

$$\Rightarrow \begin{aligned} k_1 \quad\quad + k_3 &= 0 \\ k_1+k_2 \quad\quad &= 0 \\ k_2+k_3 &= 0 \end{aligned}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1.(1)-0.(1)+1.(1)$$

$$= 1-0+1 = 2 \ ,$$

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$$|A_1| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0 \,, \quad |A_2| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \,,$$

$$|A_3| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0 \,,$$

$$k_1 = \frac{|A_1|}{|A|} = \frac{0}{2} = 0 \,, \quad k_2 = \frac{|A_2|}{|A|} = \frac{0}{2} = 0 \,, \quad k_3 = \frac{|A_3|}{|A|} = \frac{0}{2} = 0 .$$

$\Rightarrow$ S is linearly independent.

$\forall\, v = (v_1, v_2, v_3) \in R^3$, we have

$$c_1 \cdot u_1 + c_2 \cdot u_2 + c_3 \cdot u_3 = v$$

$$\Rightarrow c_1 \cdot (1, 1, 0) + c_2 \cdot (0, 1, 1) + c_3 \cdot (1, 0, 1) = (v_1, v_2, v_3)$$

$$\Rightarrow (c_1, c_1, 0) + (0, c_2, c_2) + (c_3, 0, c_3) = (v_1, v_2, v_3)$$

$$\Rightarrow (c_1 + c_3, c_1 + c_2, c_2 + c_3) = (v_1, v_2, v_3)$$

$$\Rightarrow \begin{aligned} c_1 \quad\;\; + c_3 &= v_1 \\ c_1 + c_2 \quad\;\; &= v_2 \\ c_2 + c_3 &= v_3 \end{aligned}$$

$$\Rightarrow |B| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \,,$$

$$|B_1| = \begin{vmatrix} v_1 & 0 & 1 \\ v_2 & 1 & 0 \\ v_3 & 1 & 1 \end{vmatrix} = v_1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} v_2 & 0 \\ v_3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} v_2 & 1 \\ v_3 & 1 \end{vmatrix} = v_1 + v_2 - v_3 \,,$$

$$|B_2| = \begin{vmatrix} 1 & v_1 & 1 \\ 1 & v_2 & 0 \\ 0 & v_3 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} v_2 & 0 \\ v_3 & 1 \end{vmatrix} - v_1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & v_2 \\ 0 & v_3 \end{vmatrix} = v_2 - v_1 + v_3 = -v_1 + v_2 + v_3 \,,$$

$$|B_2| = \begin{vmatrix} 1 & 0 & v_1 \\ 1 & 1 & v_2 \\ 0 & 1 & v_3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & v_2 \\ 1 & v_3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & v_2 \\ 0 & v_3 \end{vmatrix} + v_1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$
$$= v_3 - v_2 + v_1 = v_1 - v_2 + v_3$$

$$\therefore c_1 = \frac{|B_1|}{|B|} = \frac{v_1 + v_2 - v_3}{2}, \quad c_2 = \frac{|B_2|}{|B|} = \frac{-v_1 + v_2 + v_3}{2},$$
$$c_3 = \frac{|B_3|}{|B|} = \frac{-v_1 + v_2 + v_3}{2}.$$

Thus S spans R^3 over the field $(R,+,\cdot)$. Therefore S is a basis for R^3 over the field $(R,+,\cdot)$.

Definition: A nonzero vector space V over a field $(R,+,\cdot)$ is called a finite dimensional, if V contains a finite set of vectors $\{v_1, v_2, \dots, v_n\}$ that forms a basis. If no such finite set exists, then V is called infinite dimensional. The zero vector space is regarded as finite dimensional.

Theorem 5: If V is a finite dimensional vector space and $\{v_1, v_2, \dots, v_n\}$ is any basis for V, then

i- every set with more than n vectors is linearly dependent.

ii- No set with less than n vectors spans V.

44. Theorem 6: All bases for a finite dimensional vector space have the same number of vectors.

Definition: The dimension of a finite dimensional vector space V over a field $(F,+,\cdot)$ is denoted by $\dim_F V$ and is defined to be the number of vectors in a basis for V over the field $(F,+,\cdot)$. The zero vector space defined to have a zero dimension.

Examples:

1) $\dim_F F^n = n$, hence we have $\dim_R R^n = n$.

2) $\dim_C C^n = n$ and $\dim_R C^n = 2n$, and

C^n is of infinite dimension over the field $(Q,+,\cdot)$, for each $n \geq 1$.

3) R^n is of infinite dimension over the field $(Q,+,\cdot)$, for each $n \geq 1$.

4) $\dim_F M_{m\times n}(F) = mn$ where $M_{m\times n}(F)$ is the vector space of all $m\times n$ matrices over $(F,+,\cdot)$, hence we have $\dim_R M_{m\times n}(R) = mn$ and $\dim_Q M_{m\times n}(Q) = mn$.

45. Theorem 7: If V is an n-dimensional vector space over a field $(F,+,.)$ and $S=\{u_1,u_2,\dots,u_n\}\subset V$, then S is a basis for V over the field $(F,+,.)$ if either S spans V or S is linearly independent.

§6: Algebra of Subspaces

Definition: Let W_1 and W_2 be subspaces of a vector space V over a field $(F,+,.)$. The set of all vectors u which are in both W_1 and W_2 is called the intersection of the subspaces W_1 and W_2 and is denoted by $W_1\cap W_2$.

Remark: The intersection $W_1\cap W_2$ of the subspaces W_1 and W_2 of a vector space V over a field $(F,+,.)$ is never empty since the zero vector 0 belong to both W_1 and W_2.

Theorem 8: The intersection of two subspace W_1 and W_2 of a vector space V over a field $(F,+,.)$ is also a subspace of V over the field $(F,+,.)$.

46- **Proof:**

1) The zero vector 0 belongs to W_1 and W_2 which implies that $0 \in W_1 \cap W_2$. Hence $W_1 \cap W_2 \neq \Phi$.

2) $\forall\, v_1, v_2 \in W_1 \cap W_2$, we have $v_1, v_2 \in W_1$ and $v_1, v_2 \in W_2$ which implies that $v_1 + v_2 \in W_1$ and $v_1 + v_2 \in W_2$ since W_1 and W_2 are subspaces of the vector space V over the field $(F, +, \cdot)$.

Thus $v_1 + v_2 \in W_1 \cap W_2$.

3) $\forall\, r \in F$ and $\forall\, v \in W_1 \cap W_2$, we have $r \in F$ and $v \in W_1$; $r \in F$ and $v \in W_2$ which implies that $rv \in W_1$ and $rv \in W_2$ since W_1 and W_2 are subspaces of the vector space V over the field $(F, +, \cdot)$.

Thus by theorem 2, $W_1 \cap W_2$ is a subspace of the vector space V over the field $(F, +, \cdot)$.

Example: Let $W_1 = \{(x, y, z) \in R^3 \mid z = 0\}$ and $W_2 = \{(x, y, z) \in R^3 \mid y = 0\}$ be subspaces of the vector space R^3 over the field $(R, +, \cdot)$. Find the subspace $W_1 \cap W_2$.

47. *Solution:*

$(x,y,z) \in W_1 \cap W_2$

$\Rightarrow (x,y,z) \in W_1$ and $(x,y,z) \in W_2$

$\Rightarrow z=0$ and $y=0$

$\Rightarrow W_1 \cap W_2 = \{(x,0,0) \mid x \in R\}$.

Definition: Let W_1 and W_2 be subspaces of a vector space V over a field $(F,+,.)$. The subset W of V consisting of all vectors of the form $rv_1 + sv_2$ where $v_1 \in W_1$ and $v_2 \in W_2$ and $r,s \in F$ is called the sum of W_1 and W_2 and denoted by $W_1 + W_2$ (i.e. $W_1 + W_2 = \{rv_1 + sv_2 \mid v_1 \in W_1, v_2 \in W_2, r,s \in F\}$).

Theorem 9: The sum of two subspaces W_1 and W_2 of a vector space V over a field $(F,+,.)$ is also a subspace of V over the field $(F,+,.)$.

Proof: 1) The zero vector 0 belongs to W_1 and W_2 which implies that $1.0+1.0=0$ belongs to W_1+W_2 (i.e. $0 \in W_1+W_2$) i.e. $W_1+W_2 \neq \phi$.

2) $\forall (r_1v_1 + r_2v_2), (s_1w_1 + s_2w_2) \in W_1 + W_2$, (where $v_1, w_1 \in W_1$, $v_2, w_2 \in W_2$, and $r_1, r_2, s_1, s_2 \in F$), we have

48. $(r_1 v_1 + r_2 v_2) + (s_1 w_1 + s_2 w_2)$

$= (r_1 v_1 + s_1 w_1) + (r_2 v_2 + s_2 w_2)$

$= \quad u(r_1 v_1 + s_1 w_1) + \quad u(r_2 v_2 + s_2 w_2) \in W_1 + W_2$,

since $u \in F$ and $(r_1 v_1 + s_1 w_1) \in W_1$ and $(r_2 v_2 + s_2 w_2) \in W_2$ because W_1 and W_2 are subspaces of the vector space V over the field $(F, +, \cdot)$.

3) $\forall\, r \in F$ and $\forall\, (r_1 v_1 + r_2 v_2) \in W_1 + W_2$, we have $r(r_1 v_1 + r_2 v_2) = r r_1 v_1 + r r_2 v_2$ $\in W_1 + W_2$ since $r r_1, r r_2 \in F$, $v_1 \in W_1$, $v_2 \in W_2$.

Thus by theorem 2, $W_1 + W_2$ is a subspace of the vector space V over the field $(F, +, \cdot)$.

Example: Let $W_1 = \{(x, 0, 0) \mid x \in R\}$ and $W_2 = \{(0, y, 0) \mid y \in R\}$ be subspaces of the vector space R^3 over the field $(R, +, \cdot)$. Find the subspace $W_1 + W_2$.

Solution:

$W_1 + W_2 = \{r v_1 + s v_2 \mid r, s \in R \text{ and } v_1 \in W_1, v_2 \in W_2\}$

$= \{r(x, 0, 0) + s(0, y, 0) \mid r, s, x, y \in R\}$

$= \{(rx, 0, 0) + (0, sy, 0) \mid r, s, x, y \in R\}$

$= \{(rx, sy, 0) \mid r, s, x, y \in R\}$

$= \{(x_1, x_2, 0) \mid x_1, x_2 \in R\}$

49. § 7: Orthonormal Bases in R^n

Definition: A set $T=\{v_1, v_2, \dots, v_m\}$ in R^n is called orthogonal if $v_i \cdot v_j = 0$ for $i \neq j$ and $i, j = 1, 2, \dots, m$.

Definition: A set $T=\{v_1, v_2, \dots, v_m\}$ in R^n is called orthonormal if $v_i \cdot v_j = 0$ for $i \neq j$ and $i, j = 1, 2, \dots, m$; and $\|v_j\| = 1$ $\forall j = 1, \dots, m$.

Theorem 10: Let $T=\{v_1, v_2, \dots, v_m\}$ be an orthogonal set of nonzero vectors in R^n. Then T is linearly independent.

Corollary: Let $T=\{v_1, v_2, \dots, v_m\}$ be an orthonormal set of vectors in R^n is linearly independent.

Example: Let $v_1 = (0.6, 0.8, 0, 0)$, $v_2 = (-0.8, 0.6, 0, 0)$, $v_3 = (0, 0, 1, 0)$.
Prove that the set $T = \{v_1, v_2, v_3\}$ is orthonormal.

Solution:

$$v_1 \cdot v_2 = (0.6, 0.8, 0, 0) \cdot (-0.8, 0.6, 0, 0)$$
$$= 0.6 \times (-0.8) + 0.8 \times 0.6 + 0 \times 0 + 0 \times 0$$

$= -0.48 + 0.48 + 0 + 0 = 0$.

50. $v_1 \cdot v_3 = (0.6, 0.8, 0, 0) \cdot (0, 0, 1, 0)$

$= 0.6 \times 0 + 0.8 \times 0 + 0 \times 1 + 0 \times 0 = 0 + 0 + 0 + 0 = 0$.

$v_2 \cdot v_3 = (-0.8, 0.6, 0, 0) \cdot (0, 0, 1, 0)$

$= -0.8 \times 0 + 0.6 \times 0 + 0 \times 1 + 0 \times 0 = 0 + 0 + 0 + 0 = 0$.

$\therefore$ The set $T = \{v_1, v_2, v_3\}$ is an orthogonal set.

$\|v_1\| = \sqrt{(0.6)^2 + (0.8)^2 + (0)^2 + (0)^2}$

$= \sqrt{0.36 + 0.64 + 0 + 0} = \sqrt{1} = 1$

$\|v_2\| = \sqrt{(-0.8)^2 + (0.6)^2 + (0)^2 + (0)^2}$

$= \sqrt{0.64 + 0.36 + 0 + 0} = \sqrt{1} = 1$

$\|v_3\| = \sqrt{(0)^2 + (0)^2 + (1)^2 + (0)^2}$

$= \sqrt{0 + 0 + 1 + 0} = \sqrt{1} = 1$

$\therefore$ The set $T = \{v_1, v_2, v_3\}$ is an orthonormal set.

Definition: A basis $B = \{v_1, v_2, \ldots, v_n\}$ for R^n is called an orthogonal basis for R^n if B is an orthogonal set.

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Example: Let $v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $v_3 = (0, 0, 1)$. Prove that the basis $B = \{v_1, v_2, v_3\}$ is an orthonormal basis.

Solution:

$$v_1 \cdot v_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$
$$= -\frac{1}{2} + \frac{1}{2} + 0 = 0 .$$

$$v_1 \cdot v_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1) = \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0(1)$$
$$= 0 + 0 + 0 = 0 .$$

$$v_2 \cdot v_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1)$$
$$= -\frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0(1) = 0 + 0 + 0 = 0 .$$

$\therefore$ The basis $B = \{v_1, v_2, v_3\}$ is an orthogonal basis.

$$\| v_1 \| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1.$$

$$\| v_2 \| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1.$$

$$\| v_3 \| = \sqrt{(0)^2 + (0)^2 + (1)^2} = \sqrt{0 + 0 + 1} = \sqrt{1} = 1 .$$

$\therefore$ The basis $B = \{v_1, v_2, v_3\}$ is an orthonormal basis.

§8: Linear Transformation and Matrices

Definition: Let V and W be two vector spaces over the same field $(F, +, \cdot)$. A function $f : V \longrightarrow W$ is called a linear

52. transformation (or linear mapping or vector space homomorphism) if f satisfies the following:

1) $f(w+u) = f(w) + f(u) \quad \forall\, w, u \in V$

2) $f(ru) = r f(u) \quad \forall\, r \in F$ and $\forall\, u \in V$.

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Definition: Let V be a vector space over a field $(F, +, \cdot)$. A linear transformation $f : V \longrightarrow V$ is called a linear operator $\boxed{\text{on}}$ V (or linear transformation $\boxed{\text{on}}$ V).

Example: Show that the function $f : R^2 \longrightarrow R^2$ defined by
$f(x, y) = (x - y, 3x + y) \quad \forall\, (x, y) \in R^2$, is a linear transformation on the vector space R^2 over the field $(R, +, \cdot)$.

on يعني من R^2 إلى R^2 نفسه

Solution: $\forall\, (x_1, y_1), (x_2, y_2) \in R^2$, we have

$$
\begin{aligned}
& f((x_1, y_1) + (x_2, y_2)) \\
&= f(x_1 + x_2, y_1 + y_2) \\
&= ((x_1 + x_2) - (y_1 + y_2), 3(x_1 + x_2) + (y_1 + y_2)) \\
&= (x_1 + x_2 - y_1 - y_2, 3x_1 + 3x_2 + y_1 + y_2) \\
&= (x_1 - y_1 + x_2 - y_2, 3x_1 + y_1 + 3x_2 + y_2) \\
&= (x_1 - y_1, 3x_1 + y_1) + (x_2 - y_2, 3x_2 + y_2) \\
&= f(x_1, y_1) + f(x_2, y_2), \text{ and}
\end{aligned}
$$

$\forall\, r \in R$ and $\forall\, (x, y) \in R^2$, we have

53. $f(r(x,y))$
$= f(rx, ry)$
$= (rx - ry, 3(rx) + ry)$
$= (r(x-y), r(3x+y))$
$= r(x-y, 3x+y)$
$= r f(x,y)$
$\therefore$ f is a linear transformation on the vector space R^2 over the field $(R, +, \cdot)$.

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Theorem 11: If V and W be two vector spaces over a field $(F, +, \cdot)$ and $f : V \longrightarrow W$ is a linear transformation from V to W, then

i) $f(0_V) = 0_W$

ii) $f(-v) = -f(v) \quad \forall v \in V$.

Definition: If V and W be two vector spaces over a field $(F, +, \cdot)$ and $f : V \longrightarrow W$ is a linear transformation from V to W, then the kernel of f, which is denoted by $\ker f$, is the set of all $v \in V$ satisfying that $f(v) = 0_W$, i.e.
$\ker f = \{ v \in V \mid f(v) = 0_W \}$.

Definition: If V and W are two vector spaces over a field $(F,+,.)$ and $f:V\rightarrow W$ is a linear transformation from V to W, then the range of f is the set of all $f(v)$, $v\in V$ and will be denoted by Range f, i.e. Range $f=\{f(v)\mid v\in V\}$.

Theorem 12: If V and W are two vector spaces over a field $(F,+,.)$ and $f:V\rightarrow W$ is a linear transformation from V to W, then

i) The kernel of f is a subspace of V.

ii) The range of f is a subspace of W.

Example: Let $f:R^3\longrightarrow R^3$ defined by $f(x,y,z)=(x+y,2z,z)$ $\forall(x,y,z)\in R^3$ be a linear transformation on the vector space R^3 over the field $(R,+,.)$. Find $\ker f$ and Range f.

Solution:

$(x,y,z)\in\ker f$

$\Rightarrow f(x,y,z)=(0,0,0)$

$\Rightarrow (x+y,2z,z)=(0,0,0)$

$\Rightarrow x+y=0$ and $2z=0$

$\Rightarrow y=-x$ and $z=0$

Thus $\ker f=\{(x,-x,0)\mid x\in R\}$.

Range $f=\{f(x,y,z)\mid(x,y,z)\in R^3\}$

55. $= \{ (x+y, 2z, z) \mid x, y, z \in R \}$

$= \{ (u, 2z, z) \mid u, z \in R \}$.

Definition: Let f be a linear transformation on a vector space V over a field $(F, +, \cdot)$ and suppose $B = \{ v_1, v_2, \ldots, v_n \}$ is a basis of V, and let f defined on the elements of B as follows:

$$f(v_1) = k_{11} v_1 + k_{12} v_2 + \cdots + k_{1n} v_n$$
$$f(v_2) = k_{21} v_1 + k_{22} v_2 + \cdots + k_{2n} v_n$$
$$\vdots$$
$$f(v_n) = k_{n1} v_1 + k_{n2} v_2 + \cdots + k_{nn} v_n$$

Then the matrix

$$\begin{pmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & & \vdots \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{pmatrix}$$

which is denoted by $m_B(f)$ or $[f]_B$ is called the matrix representation of f relative to the basis B.

Remark: Let B be a basis of a vector space V over a field $(F, +, \cdot)$. Then any vector $v \in V$ can be written uniquely in the form $v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$ where $B = \{v_1, v_2, \ldots, v_n\}$

56. and the coordinate vector of v relative to the basis B is denoted and defined by

$$[v]_B = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = (k_1, k_2, \dots, k_n)^T$$

Example 1: Let $g: R^2 \to R^2$ defined by $g(x,y) = (x-y, 3x+y)$ $\forall (x,y) \in R^2$ be a linear operator on the vector space R^2 over the field $(R, +, \cdot)$. Find the representation matrices $m_B(g)$ and $m_S(g)$ of the linear operator g relative to each of the bases $B = \{u_1, u_2\}$ and $S = \{e_1, e_2\}$ for R^2, where $u_1 = (1,1)$, $u_2 = (-1,0)$, $e_1 = (1,0)$, $e_2 = (0,1)$. المطلوب ثاني الحل +

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Solution:

$\forall (a,b) \in R^2$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 \\ k_1 \end{pmatrix} + \begin{pmatrix} -k_2 \\ 0 \end{pmatrix}$$

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$$= \begin{pmatrix} k_1 - k_2 \\ k_1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} k_1 - k_2 &= a \\ k_1 &= b \end{aligned}$$

Thus $k_1 = b$ and $k_2 = b - a$.

57. Hence $(a,b) = b u_1 + (b-a) u_2$

Therefore $[(a,b)]_B = \begin{pmatrix} b \\ b-a \end{pmatrix}$

$g(u_1) = g(1,1) = (1-1, 3+1) = (0,4)$

$= 4u_1 + (4-0)u_2 = 4u_1 + 4u_2$ and

$g(u_2) = g(-1,0) = (-1-0, -3+0) = (-1,-3)$

$= -3u_1 + (-3-(-1))u_2 = -3u_1 - 2u_2$.

Therefore $[g]_B = m_B(g) = \begin{pmatrix} 4 & -3 \\ 4 & -2 \end{pmatrix}$.

Now $\forall (a,b) \in R^2$, we have

$g(e_1) = g(1,0) = (1-0, 3+0) = (1,3) = 1 \cdot e_1 + 3e_2$

and $g(e_2) = g(0,1) = (0-1, 0+1) = (-1,1) = -e_1 + e_2$.

Therefore $[g]_S = m_S(g) = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$.

Example 2: Let V be a vector space over a field $(F,+,\cdot)$, and let $B = \{2, t, e^{2t}, te^{2t}\}$ be a basis of V and let d be the differential operator on V, i.e. $d(v) = \frac{dv}{dt}$. Find the representation matrix of d relative to the basis B. انتبه هنا لك مشتقة لاتقع في مطلب

Solution:

$d(2) = 0 = 0 \cdot 2 + 0 \cdot t + 0 \cdot e^{2t} + 0 \cdot te^{2t}$

$d(t) = 1 = \frac{1}{2} \cdot 2 + 0 \cdot t + 0 \cdot e^{2t} + 0 \cdot te^{2t}$

$d(e^{2t}) = 2e^{2t} = 0 \cdot 2 + 0 \cdot t + 2 \cdot e^{2t} + 0 \cdot te^{2t}$

$d(te^{2t}) = e^{2t} + 2te^{2t} = 0 \cdot 2 + 0 \cdot t + 1 \cdot e^{2t} + 2 \cdot te^{2t}$

58 Thus $[d]_B = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Theorem 13: Let V be a vector space over a field $(F,+,.)$ and let B be a basis for V and let f_1 and f_2 be any linear operators on V and $k \in F$, then we have

i) $m_B(f_1+f_2) = m_B(f_1) + m_B(f_2)$ that is $[f_1+f_2]_B = [f_1]_B + [f_2]_B$.

ii) $m_B(kf_1) = k\,m_B(f_1)$ that is $[kf_1]_B = k[f_1]_B$

iii) $m_B(f_1 \circ f_2) = m_B(f_1) \cdot m_B(f_2)$ that is $[f_1 \circ f_2]_B = [f_1]_B \cdot [f_2]_B$.

Example: Let V be a vector space over a field $(F,+,.)$ and let $B = \{e^{5t}, te^{5t}\}$ and let d and h be two linear operators on V defined by $d(v) = 3\frac{d}{dt}v \quad \forall v \in V$ and $h(e^{5t}) = 3e^{5t} + te^{5t}$ and $h(te^{5t}) = e^{5t} - te^{5t}$. Find $m_B(d+3h)$ and $m_B(d \circ h)$.

Solution:

$d(e^{5t}) = 3\frac{d}{dt}e^{5t} = 3(5e^{5t}) = 15e^{5t}$

$= 15e^{5t} + 0 \cdot te^{5t}$.

$d(te^{5t}) = 3\frac{d}{dt}(te^{5t}) = 3(e^{5t} + 5te^{5t})$

$= 3e^{5t} + 15te^{5t}$.

59. Thus $m_B(d) = \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}$.

$h(e^{5t}) = 3e^{5t} + te^{5t}$
$h(te^{5t}) = e^{5t} - te^{5t}$

Thus $m_B(h) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$.

By theorem 13, $m_B(d+3h) = m_B(d) + m_B(3h)$

$= m_B(d) + 3\,m_B(h) = \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix} + 3\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix} + \begin{pmatrix} 9 & 3 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 24 & 6 \\ 3 & 12 \end{pmatrix}$, and

$m_B(d \circ h) = m_B(d) \cdot m_B(h)$

$= \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 48 & 12 \\ 15 & -15 \end{pmatrix}$