Foundation of Mathematics I

Chapter 1 Logic Theory

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Course Outline
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Contents

Chapter 1	Logic Theory	Logic, Truth Table, Tautology, Contradiction, Contingency, Rules of Proof , Logical Implication, Quantifiers, Logical Reasoning, Mathematical Proof.
Chapter 2	Sets	Definitions, Equality of Sets, Set Laws
Chapter 3	Relations on Set	Cartesian Product, Relations.
Chapter 4	Algebra of Mappings	Mappings, Types of Mappings, Composite Mapping.

References

1-Fundamental Concepts of Modern Mathematics. Max D. Larsen. 1970.
2-Introduction to Mathematical Logic, 4th edition. Elliott Mendelson.1997.
3-**اسس الرياضيات، الجزء الاول. تاليف د. هادي جابر مصطفى، رياض شاكر نعوم و نادر جورج منصور. 1980.**

4- A Mathematical Introduction to Logic, 2nd edition. Herbert B. Enderton. 2001.

THE GREEK ALPHABET

letter	name	capital
α	Alpha	Α
β	Beta	Β
γ	Gamma	Γ
δ	Delta	Δ
ε	Epsilon	Ε
ζ	Zeta	Ζ
η	Eta	Η
θ	Theta	Θ
ι	Iota	Ι
κ	Kappa	Κ
λ	Lambda	Λ
μ	Mu	Μ
ν	Nu	Ν
ξ	Xi	Ξ
ο	Omicron	Ο
π	Pi	Π
ρ	Rho	Ρ
σ ς	Sigma	Σ
τ	Tau	Τ
υ	Upsilon	Υ
φ	Phi	Φ
χ	Chi	Χ
ψ	Psi	Ψ
ω	Omega	Ω

Chapter One

Logic Theory

1.1. Logic

Definition 1.1.1.

(i) Logic is the theory of systematic reasoning and symbolic logic is the formal theory of logic.

(ii) A **logical proposition (statement or formula)** is a declarative sentence that is either true (denoted either T or 1) or false (denoted either F or 0) but not both.

(iii) The truth or falsehood of a logical proposition is called its **truth value.**

Notation: Variables are used to represent logical propositions. The most common variables used are p, q, and r.

Example 1.1.2.

$x + 2 = 2x$ when $x = -2$.

All cars are brown.

$2 \times 2 = 5$.

Here are some sentences that are not logical propositions (**paradox**).

Look out! **(Exclamatory)**

How far is it to the next town? **(Interrogative)**

$x + 2 = 2x$.

"Do you want to go to the movies?" **(Interrogative)**

"Clean up your room." **(Imperative)**

1.2. Truth Table

1.2.1. What is a Truth Table?

(i) A truth table is a tool that helps you analyze statements or arguments (defined later) in order to verify whether or not they are logical, or true.

(ii) A truth table of a logical proposition shows the condition under which the logical proposition is true and those under which it is false.

1.2.2. There are six basic operations called **connectives** that will utilize when creating a truth table. These operations are given below.

English Name	Math Name	Symbol
"and"	Conjunction	$\wedge$
"or"	Disjunction	$\vee$
"Exclusive"= "or but not both"	xor	$\underline{\vee}$
"if … then"	Implication	$\rightarrow$
"if and only if "	equivalence	$\leftrightarrow$
"not"	Negation	$\sim$

Definition 1.2.3. (Compound Statements)

If two or more logical propositions compound by connectives called compound proposition (statement). The truth value of a compound proposition depends only on the value of its components.

The rules for these connectives (operations) are as follows:

AND ($\wedge$) (conjunction): these statements are true only when both p and q are

AND ∧ (Conjunction)		
p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

OR (∨) (disjunction): these statements are false only when both p and q are false.

OR ∨ (Disjunction)		
p	q	p ∨ q
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Exclusive (⊻) one of p or q (read p or else q)

⊻ (Exclusive)		
p	q	p ⊻ q
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

If → Then Statements – These statements are false only when p is true and q is false (because anything can follow from a false premise).

If → Then		
p	q	p → q
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Here, p called **hypothesis (antecedent)** and q called **consequent (conclusion)**.

- Equivalent Forms of (**p → q)** read as:

1- If p then q":
2- p implies q
3- p is a sufficient condition for q

(Existence of H_2O is sufficient to exist of Oxygen(O))
4- p only if q= if not **q** then not **p**.
5- q if p.
6- q whenever p
7- q is a necessary condition for p.
(Existence of O is necessary to exist of H_2O)
8- q follows from p.
9- q, provided that p.

To understand why the conational statements is false only in the case when p is true but q is false considering the following example:

- Suppose your dad promises you:

"If you get an A in Foundation1, then I will buy you a laptop computer".

Here, p is "you get an **A** in Foundation1",
q is "I will buy you a notebook computer".

Then the only situation you can accuse your dad of breaking his promise is when

you get an A in Foundation1
but (and)
your dad does not buy you a notebook computer.

If you do not get an **A** in Foundtation1, then whether you dad buys you a notebook computer or not, you can't say that he breaks his promise.

- The statement $q \rightarrow p$ is called the **converse** of the statement $p \rightarrow q$ and the statement $\sim p \rightarrow \sim q$ is called the **inverse**.

For instance "**if Ali is from Baghdad then Ali is from Iraq"** is true, but the converse **"if Ali is from Iraq then Ali is from Baghdad"** may be false. The inverse "**if Ali is not from Baghdad then Ali is not from Iraq"** may be false.

- **Note** that the statements $\mathbf{p \rightarrow q}$ and $\mathbf{q \rightarrow p}$ are different.

If and only If Statements – These statements are true only when both p and q have the same truth (logical) values.

If ↔ Then		
p	q	p ↔ q
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

NOT ~ (negation) The "not" is simply the opposite or complement of its original value.

NOT ~ (negation)	
P	~p
T	**F**
F	**T**

- Note that, the negation is meaningful when used with only one logical proposition. This is not true of the other connectives.

Examples 1.2.4. Write the following statements symbolically, and then make a truth table for the statements.

(i) If I go to the mall or go to the stadium, then I will not go to the gym.

(ii) If the fish is cooked, then dinner is ready and I am hungry.

Solution.

(i) Suppose we set

p = I go to the mall

q = I go to the stadium

r = I will go to the gym

The proposition can then be expressed as "If p or q, then not r," or $(p \vee q) \rightarrow \sim r$.

p	q	r	p ∨ q	~r	(p ∨ q) →~r
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

(ii) Suppose we set

f = the fish is cooked.
r = dinner is ready.
h = I am hungry.

(a) f ⟶ (r ∧ h)

(b) (f ⟶ r) ∧ h

f	r	h	r ∧ h	f⟶(r ∧ h)	f ⟶ r	(f⟶r) ∧ h
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Exercise 1.2,5.
Build a truth table for p ⟶ (q ⟶ r) and (p ∧ q) ⟶ r.

1.3. Tautology /Contradiction / Contingency

Definition 1.3.1. (Tautology)

A tautology (theorem or lemma) is a logical proposition that is always true.

Remark 1.3.2. One informal way to check whether or not a certain logical formula is a theorem is to construct its truth table.

Example 1.3.3. $p \vee \sim p$.

Definition 1.3.4. (Contradiction)

A contradiction is a logical proposition that is always false.

Example 1.3.5. $p \wedge \sim p$.

Definition 1.3.6. (Contingency)

A contingency is a logical proposition that is neither a tautology nor a contradiction.

Example 1.3.7.

(i) The logical proposition $p \vee q \rightarrow \sim r$ is a contingency. See Example 1.2.4(i).

(ii) The logical proposition $p \vee \sim (p \wedge q)$ is a tautology.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercise 1. 1.3.8

(i) Build a truth table to verify that the logical proposition

$$(p \leftrightarrow q) \wedge (\sim p \wedge q)$$

is a contradiction.

(ii) (Low of Syllogism) Show that the logical proposition

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

is a tautology.

Definition 1.3.9. (Logically equivalent)

Propositions **r** and **s** are logically equivalents if the truth tables of **r** and **s** are the same and denoted by **r** $\equiv$ **s**.

Example 1.3.10. Show that

$$\sim(p \rightarrow q) \equiv p \wedge \sim q.$$

Solution. Show the truth values of both propositions are identical.

p	q	$\sim q$	$p \rightarrow q$	$\sim(p \rightarrow q)$	$p \wedge \sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Remark 1.3.11. (Relation Between Logical Equivalent and Tautology)

$$(r \equiv s) \equiv (r \leftrightarrow s) \text{ is a tautology.}$$

Solution.

r	s		r	s	$r \leftrightarrow s$	
T	T	**r ≡ s**	T	T	T	←
T	F		T	F	F	
F	T		F	T	F	
F	F	**r ≡ s**	F	F	T	←

From the above table of the propositions $r \equiv s$ and ($r \leftrightarrow s$ is a tautology) we get that they have the same truth table.

1.3.12. Algebra of Logical Proposition

The logical equivalences below are important equivalences that should be memorized.

1-Identity Laws: $p \wedge T \equiv p$.
$p \vee F \equiv p$.

2-Domination Laws: $p \vee T \equiv T$.
$p \wedge F \equiv F$.

3-Idempotent Laws: $p \vee p \equiv p$.
$p \wedge p \equiv p$.

4- Double Negation Law: $\sim(\sim p) \equiv p$.

5- Commutative Laws: $p \vee q \equiv q \vee p$.
$p \wedge q \equiv q \wedge p$.

6- Associative Laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

7- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

8- De Morgan's Laws: $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

9- Absorption Laws: $p \wedge (p \vee q) \equiv p$.
$p \vee (p \wedge q) \equiv p$.
$p \wedge (\sim p \vee q) \equiv p \wedge q$.
$p \vee (\sim p \wedge q) \equiv p \vee q$.

10-Implication Law: $(p \rightarrow q) \equiv (\sim p \vee q)$.
11- Contrapositive Law: $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.
12- Tautology: $p \vee \sim p \equiv T$.
13- Contradiction: $p \wedge \sim p \equiv F$.
14- Equivalence: $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$.

15- $p \underline{\vee} q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

Solution.

(8) We using truth table to prove $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

p	q	~ p	~q	p ∧ q	~ (p ∧ q)	~ p ∨ ~q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(14) We using truth table to prove $(p \to q) \wedge (q \to p) \equiv (p \leftrightarrow q)$.

p	q	p → q	q → p	p → q ∧ q → p	p ↔ q
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

(15) $p \underline{\vee} q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

p	q	p ∨ q	p ∧ q	~ (p ∧ q)	**p $\underline{\vee}$ q**	(p ∨ q) ∧ ~ (p ∧ q)
T	T	T	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F

1.4. Rules of Proof

1.4.1.

(i) Rule of Replacement.

Any term in a logical formula may be replaced be an equivalent term.

For instance, if $q \equiv r$, then $p \wedge q \equiv p \wedge r$ Rep(q:r).

(ii) Rule of Substitution.

A sentence which is obtained by substituting logical propositions for the terms of a theorem is itself a theorem.

For instance, $(p \to q) \vee w \equiv w \vee (p \to q)$ Sub(p: $p \to q$), in Commutative Law $p \vee w \equiv w \vee p$.

(iii) Rule of Inference.

1-	p $p \to q$ $\therefore q$	6-	$p \to q$ $q \to r$ $\therefore p \to r$
2-	$\sim q$ $p \to q$ $\therefore \sim p$	7-	$p \vee q$ $\sim p$ $\therefore q$
3-	p $\therefore p \vee R$	8-	$p \vee q$ $\sim p \vee r$ $\therefore q \vee r$
4-	p q $\therefore p \wedge q$	9-	$p \to q$ $r \to t$ $\therefore p \vee r \to q \vee t$
5-	$p \wedge q$ $\therefore p$	10-	p $q \to r$ $\therefore p \vee q \to p \vee r$

Example 1.4.2.

(i) Given

(1) $p \wedge q$

(2) $p \to \sim (q \wedge r)$

(3) $s \to r$

$\therefore \sim s$

Solution:

1- $p \wedge q$	1st hypothesis(premise)
2- p	Inf. (1) Properties of $\wedge$
3- q	Inf. (1) Properties of $\wedge$
4- $p \to \sim (q \wedge r)$	2nd hypothesis(premise)
5- $\sim (q \wedge r)$	Inf. (2),(4)
6- $\sim q \vee \sim r$	De Morgan's Law on (5)
7- $\sim r$	Inf. (3),(6) and Domination Laws
8- $s \to r$	3rd hypothesis(premise)
9- $\sim r \to \sim s$	Contrapositive Law
10- $\sim s$	Inf. (7),(9)

(ii) Given

(1) $\sim (p \vee q) \to r$

(2) $\sim p$

(3) $\sim r$

$\therefore q$

Solution:

1- $\sim(p \vee q) \to r$ 1st hypothesis(premise)

2- $\sim r$ 3rd hypothesis(premise)

3- $\sim r \to (p \vee q)$ Contrapositive Law and Double Negation Law

4- $p \vee q$ Inf. (2),(3)

5- $\sim p$ 2nd hypothesis(premise)

6- q Inf. (4),(5)

(iii) Given

(1) $\sim p \to (r \wedge s)$

(2) $p \to q$

(3) $\sim q$

$\therefore r$

Solution:

1- $p \to q$ 2nd hypothesis(premise)

2- $\sim q \to \sim p$ Contrapositive Law on (1)

3- $\sim q$ 3rd hypothesis(premise)

4- $\sim p$ Inf. (2),(3)

5- $\sim p \to (r \wedge s)$ 1st hypothesis(premise)

6- $r \wedge s$ Inf. (4),(5)

7- r Inf. (6) Properties of $\wedge$

(iv) Given

(1) $p \to (\sim r \wedge \sim s)$

(2) $p \vee \sim q$

(3) s

$\therefore \sim q \wedge s$

Solution:

1- $p \to (\sim r \wedge \sim s)$	1st hypothesis(premise)
2- $(r \vee s) \to \sim p$	Contrapositive Law on (1)
3- $p \vee \sim q$	2nd hypothesis(premise)
4- $\sim p \to \sim q$	Implication Law on (3)
5-$(r \vee s) \to \sim q$	Inf. (2),(4)
6- s	3rd hypothesis(premise)
7-$r \vee s$	Inf. (6)
8- $\sim q$	Inf. (5),(7)
9- $\sim q \wedge s$	Inf. (6),(8)

(v) Given

(1) $p \vee q$

(2) $q \to r$

(3) $\sim r$

$\therefore p$

Solution:

1- $q \to r$	2^{nd} hypothesis(premise)
2- $\sim r \to \sim q$	Contrapositive Law on (1)
3- $\sim r$	3^{rd} hypothesis(premise)
4- $\sim q$	Inf. (2),(3)
5- $p \vee q$	1^{st} hypothesis(premise)
6- $(p \vee q) \wedge \sim q$	Inf. (4),(5)
7- $(p \wedge \sim q) \vee (q \wedge \sim q)$	Distributive Law on (6)
8- $(p \wedge \sim q) \vee F$	Contradiction Law (7)
9- $(p \wedge \sim q)$	Identity Law on (8)
10- p	Inf. (9) properties of $\wedge$

(vi) Given

(1) "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"

(2) "If the sailing race is held, then the cup will be awarded"

(3) "The cup was not awarded"

Does this imply that: "It rained"?

Solution.

p: rain
q: foggy
r: the sailing race will be held
s: the lifesaving demonstration will go on
t: then the cup will be awarded

Symbolically, the proposition is

$$\begin{array}{ll} (1) & \sim p \vee \sim q \rightarrow r \wedge s \\ (2) & r \rightarrow t \\ (3) & \sim t \\ \hline & p \end{array}$$

1. ~t	3rd hypothesis
2. r → t	2nd hypothesis
3. ~t → ~r	Contrapositive of 2
4. ~r	Inf. (1),(3)
5. ~p∨~q → r∧s	1st hypothesis
6. ~(r∧s) → ~ (~p∨~q)	Contrapositive of 5
7. ~r ∨~s→(p∧q)	De Morgan's law and double negation law on (5)
8. ~r ∨~s	Inf. (4) and domination law
9. p ∧ q	Inf. (7),(8)
10. p	Inf. (9)

Example 1.4.3. Use the logical equivalences to show that

(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$,

(ii) $\sim(p \vee \sim(p \wedge q))$ is a contradiction,

(iii) $\sim (p \vee (\sim p \wedge q)) \equiv (\sim p \wedge \sim q)$,

(iv) $p \vee (p \wedge q) \equiv p$ (Absorption Law).

Solution.

(i) $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$ Implication Law

$\equiv \sim(\sim p) \wedge \sim q.$ De Morgan's Law
$\equiv p \wedge \sim q$ Double Negation Law

(ii) $\sim(p \vee \sim(p \wedge q))$
$\equiv \sim p \wedge \sim(\sim(p \wedge q))$ De Morgan's Law
$\equiv \sim p \wedge (p \wedge q)$ Double Negation Law
$\equiv (\sim p \wedge p) \wedge q$ Associative Law
$\equiv F \wedge q$ Contradiction Law
$\equiv F$ Domination Law and Commutative Law.

(iii) $\sim (p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ De Morgan's Law

$\equiv \sim p \wedge (p \vee \sim q)$ Double Negation Law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ Distribution Law

$\equiv (p \wedge \sim p) \vee (\sim p \wedge \sim q)$ Commutative Law

$\equiv F \vee (\sim p \wedge \sim q)$ Contradiction Law

$\equiv (\sim p \wedge \sim q) \vee F$ Commutative Law

$\equiv (\sim p \wedge \sim q)$ Identity Law

(iv) $p \vee (p \wedge q)$
$\equiv (p \wedge T) \vee (p \wedge q)$ Identity Law (in reverse)
$\equiv p \wedge (T \vee q)$ Distributive Law (in reverse)
$\equiv p \wedge T$ Domination Law
$\equiv p$ Identity Law

Example 1.4.4. Find a simple form for the negation of the proposition
"If the sun is shining, then I am going to the football game."
Solution.

p: the sun is shining q: I am going to the football game

This proposition is of the form $p \rightarrow q$. Since $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$.This is the proposition **"The sun is shining, and I am not going to the football game."**