Foundation of Mathematics 2

Chapter 1   Some Types of Functions

Dr. Bassam AL-Asadi and Dr. Emad Al-Zangana

Mustansiriyah University-College of Science-Department of Mathematics

2018-2019

Course Outline
Second Semester

Course Title: Foundation of Mathematics (2)

Code subject: 54451223

Instructors: *Mustansiriyah University-College of Science-Department of Mathematics*

Stage: The First

Contents

Chapter 1	***Some Types of Functions***	**Inverse Function and Its Properties**, **Types of Function**.
Chapter 2	***System of Numbers***	**Natural Numbers**, **Construction of Integer Numbers.**
Chapter 3	***Rational Numbers and Groups***	**Construction of Rational Numbers, Binary Operation.**

References

1-Fundamental Concepts of Modern Mathematics. Max D. Larsen. 1970.
2-Introduction to Mathematical Logic, 4th edition. Elliott Mendelson.1997.
3-**اسس الرياضيات، الجزء الثاني. تاليف د. هادي جابر مصطفى، رياض شاكر نعوم و نادر جورج منصور. 1980.**

4- A Mathematical Introduction to Logic, 2nd edition. Herbert B. Enderton. 2001.

Chapter One

Some Types of Functions

1. Inverse Function and Its Properties

We start this section by restate some basic and useful concepts.

Definition 1.1.1. (Inverse of a Relation)
Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by

$$bR^{-1}a \quad \text{if and only if} \quad aRb.$$

That is, $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

Definition 1.1.2. (Function)

(i) A relation f from A to B is said to be function iff

$$\forall x \in A\ \exists!\, y \in B \text{ such that } (x,y) \in f$$

(ii) A relation f from A to B is said to be function iff

$$\forall x \in A\ \forall y,z \in B, \text{ if } (x,y) \in f \wedge (x,z) \in f, \text{ then } y = z.$$

(iii) A relation f from A to B is said to be function iff

$$(x_1,y_1) \text{ and } (x_2,y_2) \in f \text{ such that if } x_1 = x_2, \text{ then } y_1 = y_2.$$

This property called **the well-defined relation**.

Notation 1.1.3. We write $f(a) = b$ when $(a,b) \in f$ where f is a function; that is, $(a, f(a)) \in f$. We say that b is the **image** of a under f, and a is a **preimage** of b.

Question 1.1.4. From Definition 1.1 and 1.2 that if $f : X \to Y$ is a function, does $f^{-1}: Y \to X$ exist? If Yes, does $f^{-1}: Y \to X$ is a function?

Example 1.1.5.

(i) Let $A = \{1,2,3\}$, $B = \{a,b\}$ and f_1 be a function from A to B defined bellow. $f_1 = \{(1,a),(2,a),(3,b)\}$. Then f_1^{-1} is ----------------------- .

(ii) Let $A = \{1,2,3\}$, $B = \{a, b, c, d\}$ and f_2 be a function from A to B defined bellow. $f_2 = \{(1, a), (2, b), (3, d)\}$. Then f_2^{-1} is ------------------------ .

(iii) Let $A = \{1,2,3\}$, $B = \{a, b, c, d\}$ and f_3 be a function from A to B defined bellow. $f_3 = \{(1, a), (2, b), (3, a)\}$. Then $f_3^{-1}$is ------------------------ .

(iv) Let $A = \{1,2,3\}$, $B = \{a, b, c, \}$ and f_4 be a function from A to B defined bellow. $f_4 = \{(1, a), (2, b), (3, c)\}$. Then f_4^{-1} is ------------------------ .

(v) Let $A = \{1,2,3\}$, $B = \{a, b, c, \}$ and f_5 be a relation from A to B defined bellow. $f_5 = \{(1, a), (1, b), (3, c)\}$. Then f_5 is -------------------- and f_5^{-1} is ----------------- .

Definition 1.1.6. (Inverse Function)

The function $f: X \to Y$ is said to be has inverse if the inverse relation $f^{-1}: Y \to X$ is function.

Example 1.1.7.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 3$, that is,

$$f = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x + 3\}$$

$$f = \{(x, f(x)): x \in \mathbb{R}\}$$

$$f = \{(x, x + 3) \in \mathbb{R} \times \mathbb{R}\}.$$

Then

$$f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: (y, x) \in f\}$$

$$f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = y + 3\}$$

$$f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x - 3\}$$

$$f^{-1} = \{\left(x, f^{-1}(x)\right): x \in \mathbb{R}\}$$

$$f^{-1} = \{(x, x - 3) \in \mathbb{R} \times \mathbb{R}\}.$$

That is $f^{-1}(x) = x - 3$.

f^{-1} is function as shown below.

Let $(y_1, f^{-1}(y_1))$ and $(y_2, f^{-1}(y_2)) \in f^{-1}$ such that $y_1 = y_2$, T. P. $f^{-1}(y_1) = f^{-1}(y_2)$.

Since $y_1 = y_2$, then $y_1 - 3 = y_2 - 3$ (By add -3 to both sides)

$\Longrightarrow f^{-1}(y_1) = f^{-1}(y_2)$.

(ii) $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, that is,

$$g = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2\}$$

$$g = \{\big(x, g(x)\big): x \in \mathbb{R}\}$$

$$g = \{(x, x^2) \in \mathbb{R} \times \mathbb{R}\}.$$

Then

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: (y, x) \in g\}$$

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = y^2\}$$

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \pm\sqrt{x}\}$$

$$g^{-1} = \big\{\big(x, \pm\sqrt{x}\big) \in \mathbb{R} \times \mathbb{R}\big\}, \text{ that is } g^{-1}(x) = \pm\sqrt{x}.$$

g^{-1} is not function since $g^{-1}(4) = \pm 2$.

Remark: If f is a function, then $f(x)$ is always is an element in the $Ran(f)$ for all x in $Dom(f)$ but $f^{-1}(y)$ may be a subset of $Dom(f)$ for all y in $Cod(f)$.

Theorem 1.1.8. Let $f : A \to B$ be a function. Then f is bijective iff the inverse relation f^{-1} is a function from B to A.

Proof.
Suppose $f : A \to B$ is bijective. To prove f^{-1} is a function from B to A.
$f^{-1} \neq \emptyset$ since f is onto.
$(*)$ Let (y_1, x_1) and $(y_2, x_2) \in f^{-1}$ such that $y_1 = y_2$, to prove $x_1 = x_2$.

(x_1, y_1) and $(x_2, y_2) \in f$ Def. of f^{-1}

(x_1, y_1) and $(x_2, y_1) \in f$ By hypothesis $(*)$

$x_1 = x_2$ Def. of 1-1 on f

$\therefore f^{-1}$ is a function from B to A.

Conversely, suppose f^{-1} is a function from B to A, to prove $f : A \to B$is bijective, that is, 1-1 and onto.

1-1: Let $a, b \in X$ and $f(a) = f(b)$. To prove $a = b$.

$(a, f(a))$ and $(b, f(b)) \in f$ Hypothesis ($f$is function)

$(a, f(a))$ and $(b, f(a)) \in f$ Hypothesis ($f(a) = f(b)$)

$(f(a), a)$ and $(f(a), b) \in f^{-1}$ Def. of inverse relation f^{-1}

$a = b$ Since f^{-1} is function

$\therefore f$ is 1-1.

onto: Let $b \in Y$. To prove $\exists a \in A$ such that $f(a) = b$.

$(b, f^{-1}(b)) \in f^{-1}$ Hypothesis (f^{-1} is a function from B to A)

$(f^{-1}(b), b) \in f$ Def. of inverse relation f^{-1}

Put $a = f^{-1}(b)$.

$a \in A$ and $f(a) = b$ Hypothesis (f is function)

$\therefore f$ is onto.

Definition 1.1.9. Let $f : X \to Y$ be a function and $A \subseteq X$ and $B \subseteq y$.

(i) The set $f(A) = \{f(x) \in Y: x \in A\} = \{y \in Y: \exists x \in A$ such that $y = f(x)\}$ is called the **direct image of A by f**.

(ii) The set $f^{-1}(B) = \{x \in X: f(x) \in B\} = \{x \in X: \exists y \in B$ such that $f(x) = y\}$ is called the **inverse image of B with respect to f**.

Remark: Let $f : X \to Y$ be a function and $A \subseteq X$. If then $y \in f(A)$, then $f^{-1}(y) \subseteq A$.

Example 1.1.10.

(i) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 - 1$. $f^{-1}(15) = \{x \in \mathbb{R}: x^4 - 1 = 15\}$

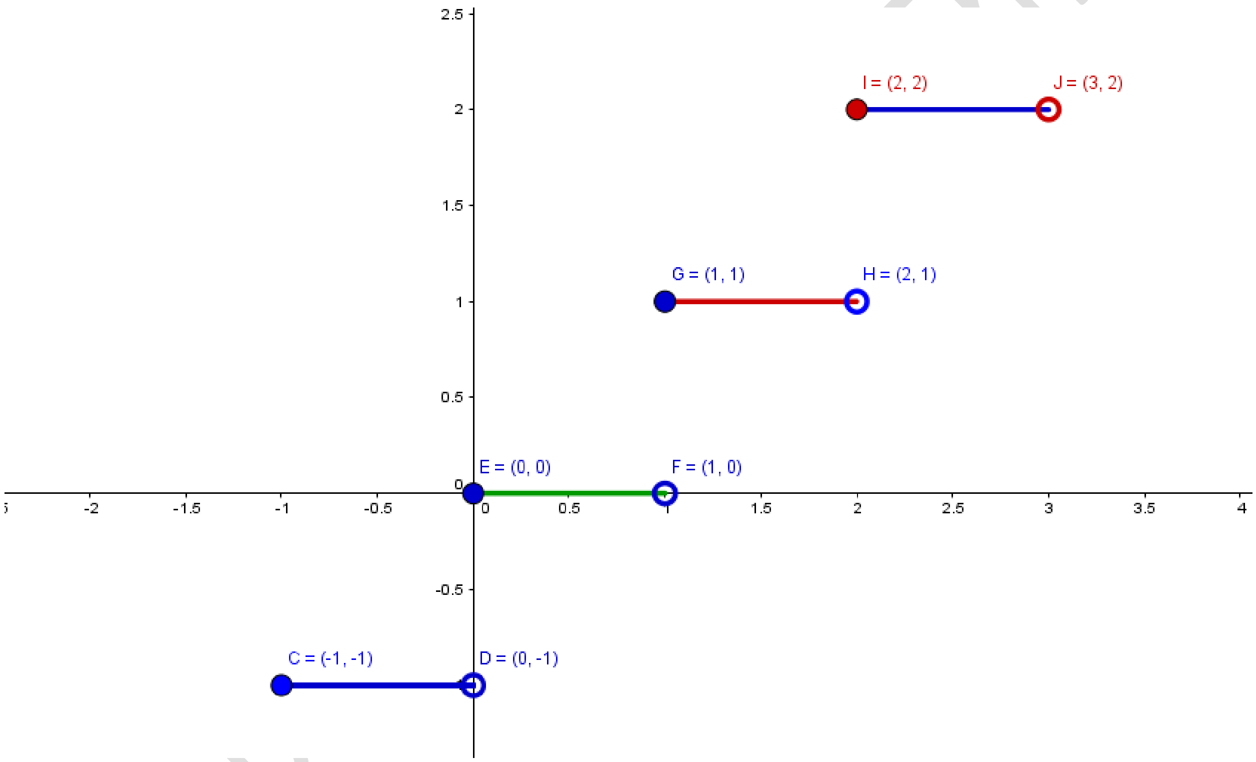
$$= \{x \in \mathbb{R}: x^4 = 16\} = \{-2,2\}.$$

(ii) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \end{cases}$.

$D(f) = [-1,3), R(f) = \{-1,0,1,2\}.$

$f([-1,-1/2]) = -1.$ $f([-1,0]) = \{-1,0\}.$

$f^{-1}(0) = [0,1).$ $f^{-1}([1,3/2]) = [1,2).$



Definition 1.1.11.

(i) A function $I_A : A \to A$ defined by $I_A(x) = x$, for every $x \in A$ is called the **identity** function on A. $I_A = \{(x,x): x \in A\}$.

(ii) Let $A \subseteq X$. A function $i_A : A \to X$ defined by $i_A(x) = x$, for every $x \in A$ is called the **inclusion** function on A.

Theorem 1.1.12.

If $f : X \to Y$ is a bijective function, then $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$.

Proof. Exercise.

Example 1.1.13. Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be a function defined as

$$f(m,n) = (m+n, m+2n).$$

f is bijective(**Exercise**).

To find the inverse f^{-1} formula, let $f(n,m) = (x,y)$. Then

$(m+n, m+2n) = (x,y)$. So, the we get the following system

$$\begin{aligned} m+n &= x \ldots . (1) \\ m+2n &= y \ldots . (2) \end{aligned}$$

From (1) we get $m = x - n \quad \ldots . (3)$

$n = y - x$ Inf (2) and (3) $\ldots . (4)$

$m = 2x - y$ Rep $(n: y - x)$ or sub(4) in (3)

Define f^{-1} as follows

$$f^{-1}(x,y) = (2x - y, y - x).$$

We can check our work by confirming that $f \circ f^{-1} = I_Y$.

$$\begin{aligned} (f \circ f^{-1})(x,y) &= f(2x - y, y - x) \\ &= ((2x - y) + (y - x), (2x - y) + 2(y - x)) \\ &= (x, 2x - y + 2y - 2x) = (x,y) = I_Y(x,y) \end{aligned}$$

Remark 1.1.14. If $f : X \to Y$ is oneto-one but not onto, then one can still define an inverse function $f^{-1} : R(f) \to X$ whose domain in the range of f.

Theorem 1.1.15. Let $f : X \to Y$ be a function.

(i) If $\{Y_j \subset Y : j \in J\}$ is a collection of subsets of Y, then

$f^{-1}(\bigcup_{j \in J} Y_j) = \bigcup_{j \in J} f^{-1}(Y_j)$ and $f^{-1}\left(\bigcap_{j \in J} Y_j\right) = \bigcap_{j \in J} f^{-1}(Y_j)$

(ii) If $\{X_i \subset X: i \in I\}$ is a collection of subsets of X, then

$f(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} f(X_i)$ and $f(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} f(X_i)$.

(iii) If A and B are subsets of X such that $A = B$, then $f(A) = f(B)$. The converse is not true.

(iv) If C and D are subsets of Y such that $C = D$, then $f^{-1}(C) = f^{-1}(D)$. The converse is not true.

(v) If A and B are subsets of X, then $f(A) - f(B) \subseteq f(A - B)$. The converse is not true.

(vi) If C and D are subsets of Y, then $f^{-1}(C) - f^{-1}(D) = f^{-1}(C - D)$.

Proof:

(i) Let $x \in f^{-1}(\bigcup_{j\in J} Y_j)$.

$\exists y \in \bigcup_{j\in J} Y_j$ such that $f(x) = y$ Def. of inverse image

$y \in Y_j$ for some $j \in J$ ($f(x) \in Y_j$ for some $j \in J$) Def. of $\cup$

$x \in f^{-1}(Y_j)$ Def. of inverse image

so $x \in \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\cup$

It follow that $f^{-1}(\bigcup_{j\in J} Y_j) \subseteq \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\subseteq$ $\ldots\ldots(*)$

Conversely, If $x \in \bigcup_{j\in J} f^{-1}(Y_j)$, then $x \in f^{-1}(Y_j)$, for some $j \in J$ Def. of $\cup$

So $f(x) \in Y_j$ and $f(x) \in \bigcup_{j\in J} Y_j$ Def. of inverse and $\cup$

$x \in f^{-1}(\bigcup_{j\in J} Y_j)$ Def. of inverse f^{-1}

It follow that $\bigcup_{j\in J} f^{-1}(Y_j) \subseteq f^{-1}(\bigcup_{j\in J} Y_j)$ Def. of $\subseteq$ $\ldots\ldots(**)$

$f^{-1}(\bigcup_{j\in J} Y_j) = \bigcup_{j\in J} f^{-1}(Y_j)$ From $(*), (**)$ and Def. of $=$

Example 1.1.16. Let $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function defined as $f(x) = 1$.

$\mathbb{Z}_e \cap \mathbb{Z}_o = \emptyset$. $f(\mathbb{Z}_e \cap \mathbb{Z}_o) = f(\emptyset) = \emptyset$. But $f(\mathbb{Z}_e) \cap f(\mathbb{Z}_o) = \{1\}$.

2.Types of Function

Definitions 1.2.1.

(i) (Constant Function)

The function $f: X \longrightarrow Y$ is said to be **constant function** if there exist a unique element $b \in Y$ such that $f(x) = b$ for all $x \in X$.

(ii) (Restriction Function)

Let $f: X \longrightarrow Y$ be a function and $A \subseteq X$. Then the function $g: A \longrightarrow Y$ defined by $g(x) = f(x)$ all $x \in X$ is said to be **restriction function** of f and denoted by $g = f|_A$.

(iii) (Extension Function)

Let $f: A \longrightarrow B$ be a function and $A \subseteq X$. Then the function $g: X \longrightarrow B$ defined by $g(x) = f(x)$ all $x \in A$ is said to be **extension function** of f from A to X.

(iv) (Absolute Value Function)

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ which defined as follows

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$$

is called the **absolute value function.**

(v) (Permutation Function)

Every bijection function f on a non empty set A is said to be **permutation** on A.

(vi) (Sequence)

Let A be a non empty set. A function $f: \mathbb{N} \longrightarrow A$ is called a sequence in A and denoted by $\{f_n\}$, where $f_n = f(n)$.

(vii) (Canonical Function)

Let A be a non empty set, R an equivalence relation on A and A/R be the set of all equivalence class. The function $\pi: A \longrightarrow A/R$ defined by $\pi(x) = [x]$ is called the **canonical function**.

(viii) (Projection Function)

Let A_1, A_2 be two sets. The function $P_1: A_1 \times A_2 \longrightarrow A_1$ defined by $P_1(x, y) = x$ for all $(x, y) \in A_1 \times A_2$ is called the **first projection.**

The function $P_2: A_1 \times A_2 \longrightarrow A_2$ defined by $P_2(x, y) = y$ for all $(x, y) \in A_1 \times A_2$ is called the **second projection.**

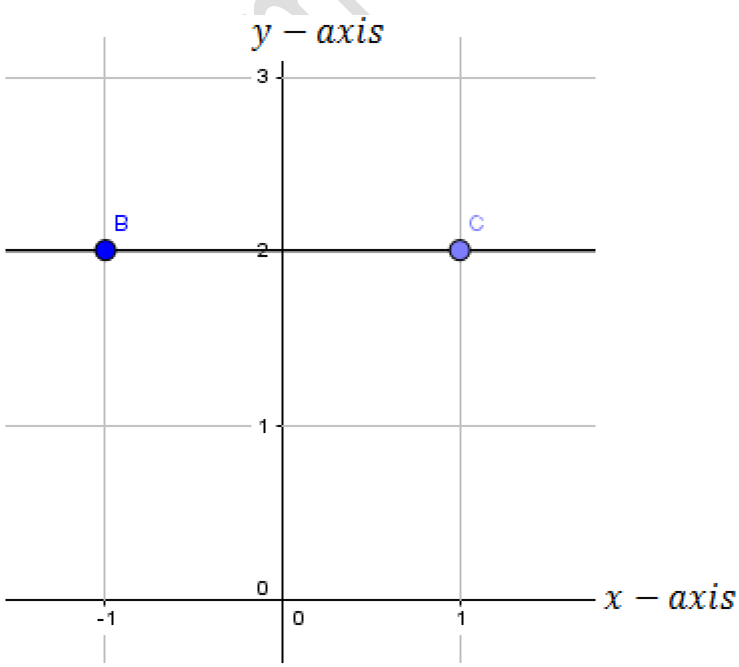
(ix) (Cross Product of Functions)

Let $f: A_1 \longrightarrow A_2$ and $g: B_1 \longrightarrow B_2$ be two functions. The cross product of f with g, $f \times g: A_1 \times B_1 \longrightarrow A_2 \times B_2$ is the function defined as follows:

$$(f \times g)(x, y) = (f(x), g(y)) \text{ for all } (x, y) \in A_1 \times B_1.$$

Examples 1.2.2.

(i)(Constant Function). $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2, \forall x \in \mathbb{R}$. $D(f) = \mathbb{R}$, $R(f) = \{2\}$, $Cod(f) = \mathbb{R}$.

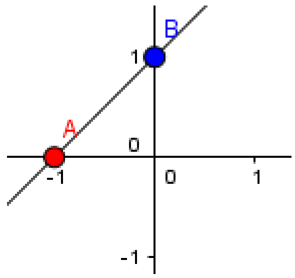


(ii) (Restriction Function). $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x + 1, \forall x \in \mathbb{R}$.

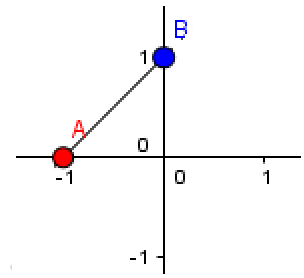
$D(f) = \mathbb{R}$, $R(f) = \mathbb{R}$, $Cod(f) = \mathbb{R}$. Let $A = [-1,0]$.

$g = f|_A : A \longrightarrow \mathbb{R}$. $g(x) = f(x) = x + 1, \forall x \in A$.

$D(g) = A$, $R(g) = [0,1]$, $Cod(g) = \mathbb{R}$.



$f(x) = x + 1$



$g = f|_A$

(iii) (Extension Function). $f: [-1,0] \longrightarrow \mathbb{R}$, $f(x) = x + 1, \forall x \in [-1,0]$.

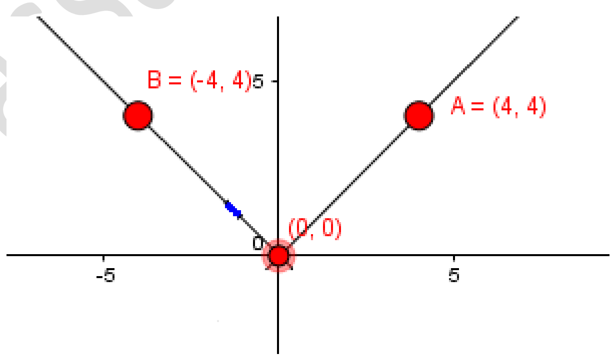
$D(f) = [-1,0]$, $R(f) = [0,1]$, $Cod(f) = \mathbb{R}$.

Let $A = \mathbb{R}$. $g: A \longrightarrow \mathbb{R}$. $g(x) = f(x) = x + 1, \forall x \in A$.

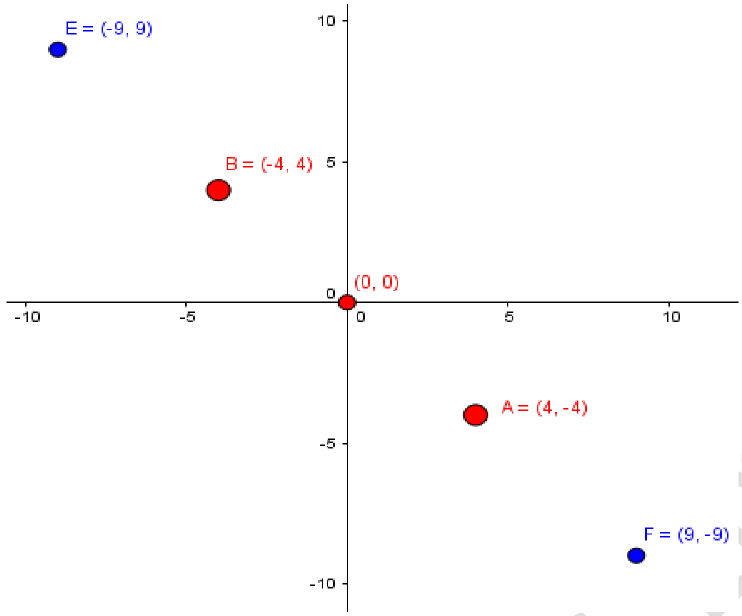
$D(g) = A$, $R(g) = \mathbb{R}$, $Cod(g) = \mathbb{R}$.

(iv) (Absolute Value Function) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$.

$D(f) = \mathbb{R}$, , $R(f) = [0, \infty)$, $Cod(f) = \mathbb{R}$.



(v) (Permutation Function). $f: \mathbb{N} \longrightarrow \mathbb{N}$, $f(x) = -x, \forall x \in \mathbb{N}$. The function is bijective, so it is permutation function. $D(f) = \mathbb{N}$, , $R(f) = \mathbb{N}$, $Cod(f) = \mathbb{N}$.



(vi) (Sequence). $f: \mathbb{N} \longrightarrow \mathbb{Q}$, $f(n) = \frac{1}{n}$, $\forall x \in \mathbb{N}$. $\{f_n\} = \{\frac{1}{n}\}_{n=1}^{\infty}$.

(vii) (Canonical Function). Let R be an equivalence relation defined on $\mathbb{Z}$ as follows:

xRy iff $x - y$ is even integer, that is, $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}: x - y \text{ even}\}$.

$[0] = \{x \in \mathbb{Z}: x - 0 \text{ even}\} = \{\dots, -4, -2, 0, 2, 4, \dots\} = [2] = [-2] = \cdots$.

$[1] = \{x \in \mathbb{Z}: x - 1 \text{ even}\} = \{\dots, -5, -3, -1, 1, 3, 5, \dots\} = [-1] = [3] = \cdots$.

$\mathbb{Z}/R = \{[0], [1]\}$.

$\pi(0) = [0] = \pi(2) = \pi(-2) = \cdots$.

$\pi(1) = [1] = \pi(-1) = \pi(-3) = \cdots$.

(viii) (Projection Function)

$P_1: \mathbb{Z} \times \mathbb{Q} \longrightarrow \mathbb{Z}$, $P_1(x, y) = x$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Q}$. $P_1\left(2, \frac{2}{5}\right) = 2$. $P_1(\mathbb{Z}, \frac{2}{5}) = \mathbb{Z}$.

$P_1^{-1}(3) = \{3\} \times \mathbb{Q}$.

(ix) (Cross Product of Functions)

$f: \mathbb{N} \longrightarrow \mathbb{Q}$, $f(n) = \frac{1}{n}, \forall n \in \mathbb{N}$ and $f: \mathbb{N} \longrightarrow \mathbb{N}$, $f(x) = -x, \forall x \in \mathbb{N}$

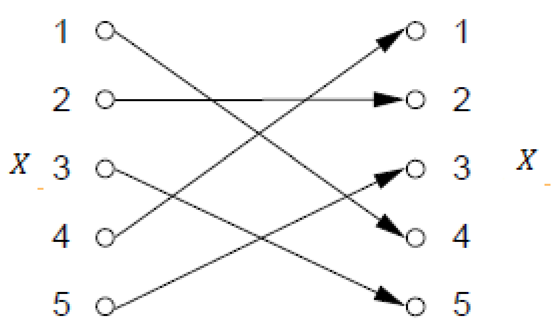
$f \times g: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q} \times \mathbb{N}, (f \times g)(x, y) = (f(x), g(y))$

$= (\frac{1}{x}, -y)$ for all $(x, y) \in \mathbb{N} \times \mathbb{N}$.

(iix) (Involution Function)
Let X be a finite set and let f be a bijection from X to X (that is, $f: X \longrightarrow X$).
The function f is called an ***involution*** if $\boxed{f = f^{-1}}$. An equivalent way of stating this is

$$\boxed{f(f(x)) = x \quad \text{for all} \quad x \in X.}$$

The figure below is an example of an involution on a set X of five elements. In the diagram of an involution, note that if j is the image of i then i is the image of j.



Exercise 1.2.3.

(i) Let R be an equivalence relation defined on $\mathbb{N}$ as follows:

$$R = \{(x, y) \in \mathbb{N} \times \mathbb{N}: x - y \text{ divisble by } 3\}.$$

1- Find $\mathbb{N}/R$. **2-** Find $\pi([0]), \pi([1]), \pi^{-1}([2])$.

(ii) Prove that the Projection function is onto but not injective.

(iii) Prove that the Identity function is bijective.

(iv) Prove that the inclusion function is bijective onto its image.

(v) Let $f: A_1 \longrightarrow A_2$ and $g: B_1 \longrightarrow B_2$ be two functions. If f and g are both 1-1 (onto), then, $f \times g$ is 1-1(onto).

(vi) If $f: X \longrightarrow Y$ is a bijective function, then f^{-1} is bijective function.

(vii) If $f: X \longrightarrow Y$ is a bijective function, then

1- $f \circ f^{-1} = I_Y$ is bijective function. **2-** $f^{-1} \circ f = I_X$ is bijective function.

(viii) Let $f: X \longrightarrow Y$ and If $g: Y \longrightarrow X$ are functions. If $g \circ f = I_X$, then f is injective and g is onto.

(ix) Let $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined as follows:

$$f(x, y) = x^2 + y^2.$$

1- Find the $f(\mathbb{R} \times \mathbb{R})$ (image of f).

2- Find $f^{-1}([0,1])$.

3- Does f 1-1 or onto?

4- Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = \sqrt{2 - y^2}\}$. Find $f(A)$.