**14. Compactness**

 (14.1)**Definition**: Let $\{A\_{λ}\}\_{λ\in Λ}$ is a family of subsets in $X$, and let $A⊆X$. We said that $F$ is a covering of $A$, if $A⊆A\_{λ}$. If $Λ$ is a finite, then $F$ is a finite covering of $A$.

(14.2)**Example:** Let $X=\left\{1,2,3,4,5\right\}, A=\{1,2\}$, then

1. A family$ \{\left\{1\right\},\left\{1,2\right\}\}$ represents a covering of $A$, since $\left\{1,2,3\right\}=\left\{1\right\}∪\{2,3\}⟹A⊆\left\{1\right\}∪\{2,3\}$.
2. A family $\{\left\{2\right\},\left\{4,5\right\}\}$ does not represent a covering of $A$, since $A⊈\{\left\{2\right\}∪,\left\{4,5\right\}\}$.
3. A family $\{\left\{1,2\right\},\left\{3,4\right\},\left\{1,3,5\right\}\}$ represents a covering of $A$ and $X$.

(14.3)**Example:**

1. A family $F=\{\left[\frac{1}{n},1-\frac{1}{n}\right]:n\in Z^{+}\}$ represents an infinite covering of $A=(0,1)$.
2. A family $F=\{(n,n+3):n\in Z\}$ represents an infinite covering of $R$.
3. A family $F=\{(n,n+1):n\in Z\}$ does not represent a covering of $R$.

(14.4)**Definition**: Let $A⊆X,$ $ F=\{A\_{λ}\}\_{λ\in Λ}, G=\{B\_{γ}\}\_{γ\in Λ^{'}}$ are covering of $A$, we said that $F$ is a sub covering from $G$, if for all $λ\in Λ ∃γ\in Λ^{'}\ni A\_{λ}=B\_{γ}$.

(14.5)**Example:** Each of $F=\left\{\left(n,n+3\right):n\in Z\right\},G=\{(r,r+3):r\in R\} $are covering of $R$ and $F$ is a subfamily of $G$.

(14.6)**Definition**: Let $A$ is a subset of $(X,d)$ and let $ F=\{A\_{λ}\}\_{λ\in Λ}$ is a covering of $A$. We said that $F$ is an open cover, if $A\_{λ}$ is an open set in $X ∀λ\in Λ$.

(14.7)**Example**: In $(R,d\_{u})$. Prove that a family $F=\{\left(\frac{1}{n},2\right):n\in Z^{+}\}$ is an open cover of $A=(0,1)$.

**Solution:** let $x\in A⟹0<x<1$,

Since $x>0$ (by Archimedes property)$ ⟹∃k\in Z^{+}\ni \frac{1}{k}<x$.

Since $x<1⟹\frac{1}{k}<x<2⟹x\in (\frac{1}{k},2)⟹x\in \bigcup\_{n\in Z^{+}}^{}(\frac{1}{n},2)$

$⟹A⊂\bigcup\_{n\in Z^{+}}^{}(\frac{1}{n},2)⟹F$ is a covering of $A$.

Since $(\frac{1}{n},2)$ is open set $∀n\in Z^{+}⟹F$ is an open set of $A$.

(14.8)**Example**: In $(R,d\_{u})$, we have

$F\_{1}=\{(-n,n):n\in Z^{+}\}$, $F\_{2}=\{(-3n,3n):n\in Z^{+}\}$,$F\_{3}=\{(2n-1,2n+1):n\in Z^{+}\}$ are an open cover of $R$, also $F\_{2}$ is a sub cover of$ F\_{1}$.

(14.9)**Example**: Let $(X,d)$ be discrete metric space and $A⊆X$. Prove that$ F=\{\left\{x\right\}:x\in A\}$ is an open cover of $A$.

**Solution:** since $A=\bigcup\_{x\in A}^{}\{x\}⟹F$ is a covering of $A$.

Since $\left(X,d\right)$ is discrete metric space $⟹\left\{x\right\}$ an open set in $X ∀x\in X$

$⟹F$ is an open cover of $A$.

(14.10)**Definition**: Let $(X,d)$ is metric space and let $A⊆X$. We said that $A$ is a compact set in $X$, if for all open cover $A$ contains a finite sub covering.

(14.11)**Example**: In $(R,d\_{u})$, we have

1. $A=(0,1)$does not compact in $R$.
2. $A=\{1,\frac{1}{2}, \frac{1}{3},…, \frac{1}{n},…,0\}$ is a compact in $R$.
3. A space $R$ does not compact.

**Solution:** (1) Take $F=\{(\frac{1}{n},2):n\in Z^{+}\}$ is an open cover of $A$, but $F$ does not contain on a finite sub cover$ ⟹A$ does not compact.

(14.12) **Example**: Every indiscrete metric space is a compact, since an unique open cover of $X$ is $\{X\}$.

(14.13) **Theorem:** Every finite set in a metric space is a compact.

**Proof:** let $A$ is a finite set in $\left(X,d\right)⟹A=\{a\_{1}, a\_{2}, …,a\_{n} \}$

Let $ F=\bigcup\_{λ\in Λ}^{}G\_{λ}$ is a open cover of $A$ in $X$.

$⟹A⊆\bigcup\_{λ\in Λ}^{}G\_{λ}, G\_{λ}$is an open set in $X ∀λ\in Λ$.

Since $a\_{i}\in A ∀i=1,2,…,n$

$$⟹a\_{i}\in \bigcup\_{λ\in Λ}^{}G\_{λ} ∀i=1,2,…,n$$

$$⟹∀i ∃ λ\_{i}\in Λ\ni a\_{i}\in G\_{λ\_{i}}$$

$⟹\{G\_{λ\_{1}},G\_{λ\_{2}},…,G\_{λ\_{n}}\}$ is a finite sub covering from $F$ of $A$.

$⟹A $is a compact set.

(14.14)**Example**: Let $(X,d)$ is discrete metric space, then $X$ is a compact $⟺$ $X$ is a finite.

(14.15)**Theorem**: Let $(Y,d\_{Y})$ is a subspace of a metric space $(X,d)$ and $A⊂Y$, then $A$ is a compact in $X⟺A$ is a compact in $Y$.

(14.16)**Theorem**: Every closed set in a compact metric space is a compact.

(14.17)**Theorem**: Every compact set in a metric space is a closed and bounded.

(14.18)**Definition:** We said that a family of sets that satisfy a finite intersection property, if intersection every finite subfamily is a non- empty set.

 (14.19)**Theorem;** A metric space $(X,d)$ is a compact $⟺$ if every family of a closed sets satisfies a finite intersection property, then its non-empty set.

(14.20)**Definition:** We said that a metric space $(X,d)$ is a countable compact, if for all open cover and countable in $X$ contains on a finite sub covering.

(14.21)**Theorem;** A metric space $(X,d)$ is a countable compact $⟺$ every countable family of a closed sets and satisfy a finite intersection property is a non- empty intersection.