**3. Sequences**

(3.1) **Definition**: Let $X$be a non-empty set. A function which its domain $N$ and its codomain $X$ is called a sequence in$ X$, such that if $f:N⟶X, ∀n\in N ∃ x\_{n}\in X\ni f\left(n\right)=x\_{n}$.

(3.2) **Example:** If $\{x\_{n}\}$ be a sequence defined in $R\ni x\_{n}=\left(-1)^{n} ∀n\in N⟹\{x\_{n}\right\}=\{\left(-1)^{n}\right\}=\{1,-1,1,-1,…\}$ a range is $\{x\_{n}:n\in N\}=\{-1,1\}$.

(3.3) **Definition**: Let $\{x\_{n}\}, \{y\_{n}\}$ be a sequences in $X$, we say that $\{y\_{n}\}$ is a subsequence of $\{x\_{n}\}$, if there is a function $φ:N⟶N\ni $

1. $x\_{n}∘φ=y\_{n}$;
2. $∀n\in N ∃k\in N\ni φ\left(m\right)\geq n ∀m\geq k$.

(3.4) **Example:** Let $x\_{n}=\frac{1}{n} , σ\_{n}=\frac{1}{2n-1}$, we note that $\{σ\_{n}\}$ is a subsequence of $\{x\_{n}\}$, since if we define $ψ:N⟶N$ by$ ψ\left(n\right)=2n-1$, $σ\_{n}=x\_{n}∘ψ=\frac{1}{2n-1}$ and then $\{\frac{1}{2},\frac{1}{3},\frac{1}{4},…\}$ be a subsequence of {$\frac{1}{n}\}$.

(3.5) **Note:** If $\{y\_{n}\}$ is a subsequence of $\{x\_{n}\}$ and $\{y\_{n}\}$ is a subsequence of$ \{z\_{n}\}$, then $\{z\_{n}\}$ is a subsequence of $\{x\_{n}\}$.

(3.6) **Definition**: If $\{x\_{n}\}$ be a sequence in a partially ordered set $X$, we say that $\{x\_{n}\}$ be an increasing, if $x\_{n}\leq x\_{n+1}∀ n$, and we say that $\{x\_{n}\}$ be a decreasing, if $x\_{n+1}\leq x\_{n}∀ n$ and we say that $\{x\_{n}\}$ be a monotone, if $\{x\_{n}\}$ an increasing or a decreasing.

(3.7) **Note:**

* $x\_{n}\uparrow = \{x\_{n}\}$ be an increasing.
* $x\_{n}\uparrow x= \{x\_{n}\}$ be an increasing and $ x\_{n}=$ sup $x\_{n}$, $n\in N$.
* $x\_{n}\downright = \{x\_{n}\}$ be a decreasing.
* $x\_{n}\downright x = \{x\_{n}\}$ be a decreasing and $ x\_{n}=$inf $x\_{n}$, $n\in N$.

(3.8) **Definition**: Let $\{x\_{n}\}$ be a sequence in a partially ordered set $X$, we say that $\{x\_{n}\}$ is a converges to $x\in X$, if there is $\{a\_{n}\}, \{b\_{n}\}$ in $X$, such that

1. $a\_{n}\leq x\_{n}\leq b\_{n} ∀n$;
2. $a\_{n}\uparrow x$ and $x\_{n}\downright x\_{n}$.

(3.9) **Note:** $x$ is called a converge point and written$ x\_{n}$ $→x$.

(3.10) **Definition**: Let $\{x\_{n}\}$ be a sequence in a partially ordered set $X$, we have

* Inferior limit $=$ lim inf $x\_{n}$, where lim inf $x\_{j}, n\in N, j\geq n$.
* Superior limit $=$ lim sup $x\_{n}$, where lim sup $x\_{j}, n\in N, j\geq n$.

(3.11) **Note:** If lim sup $x\_{n}=$ lim sup $x\_{n}=x⟹ x\_{n}$ $→x$.

**Real Sequences**

(3.12) **Note:** We saythat $\{x\_{n}\}$ be a real sequence if $X=R$.

(3.13) **Definition**: Arithmetic progression is a sequence which be subtract output of every term from direct previous term is equal to constant called progression basis and denoted by $d$.

(3.14) **Example:** Arithmetic progression which its first term $a$ and its basis $d$ is

$\{a,a+d,a+2d,…,a+\left(n-1\right)d,…\}$. The general term of arithmetic progression $\{x\_{n}\}$ is $x\_{n}=a+\left(n-1\right)d$ where $a$ represents a first term and $d$ represents a basis with the partial summation

 $S\_{n}=\sum\_{k=1}^{n}x\_{k}=\sum\_{k=1}^{n}(a+\left(k-1\right)d)=\frac{n}{2}(2a+(n-1))$.

(3.15) **Definition**: Geometry progression is a sequence which output of division of every term on direct previous term is equal to a constant called progression basis and denoted by $r$.

(3.16) **Example:** Geometric progression which its first term $a$ and its basis $r$ is

$\{r,ar,ar^{2},…,ar^{n},…\}$. The general term is $x\_{n}=ax^{n-1}$ where $a$ represents a first term and $r$ represents a basis with the partial summation

 $S\_{n}=\sum\_{k=1}^{n}x\_{k}=\sum\_{k=1}^{n}ar^{k-1}=\frac{a(1-r^{n})}{1-r},r\ne 1$.

If $r=1⟹S\_{n}=a+a+…+a=na$.

If $\left|r\right|<1⟹\sum\_{k=1}^{\infty }ar^{n-1}=\frac{a}{1-r}$ .

(3.17)**Definition**: Arithmetic geometric progression is $\{a,(a+d)r,(a+2d)r^{2},…,(a+\left(n-1\right)d)r^{n-1},…\}$. The general term is $x\_{n}=(a+\left(n-1\right)d)r^{n-1}$ and the partial summation is

 $S\_{n}=\sum\_{k=1}^{n}x\_{k}=\sum\_{k=1}^{n}(a+\left(k-1\right)d)r^{k-1}=\frac{a(1-r^{n})}{1-r}+\frac{rd\left(1-nr^{n-1}\right)+(n-1)r^{n}}{(1-r)^{2}},r\ne 1$.

If $\left|r\right|<1⟹\sum\_{k=1}^{\infty }(a+\left(n-1\right)d)r^{n-1}=\frac{a}{1-r}+\frac{rd}{(1-r)^{2}}$ .

(3.18) **Definition**: Let $\{x\_{n}\}$ be a real sequence, we say that $\{x\_{n}\}$

1. Convergent, if $∃ r\in R\ni ∀ε>0 ∃k\in Z^{+}\ni \left|x\_{n}-x\right|<ε ∀n>k$, we say that a point $x$ is a limit point of $\{x\_{n}\}$ and its written by $\lim\_{n\to \infty }x\_{n}$ or$ x\_{n}\rightarrow x$ where $n\rightarrow \infty $, therefore $ x\_{n}\rightarrow x$ iff $\left|x\_{n}-x\right|\rightarrow 0$.
2. Divergent, if $\{x\_{n}\}$ does not convergent.
3. Cauchy sequence, if $∀ε>0∃ k\in Z^{+}\ni \left|x\_{n}-x\_{m}\right|<ε ∀n,m>k $and then $\{x\_{n}\}$ is a Cauchy sequence iff $\left|x\_{n}-x\_{m}\right|\rightarrow 0$ where $n,m\rightarrow \infty $.

(3.19) **Examples:**

1. Show that $\{x\_{n}\}\rightarrow x$.

**Solution:** since $∀ε>0⟹\left|x\_{n}-x\right|=0<ε$.

1. Show that $\{\frac{1}{n}\}\rightarrow 0$.

**Solution:** since $∀ε>0$ (by Archimedes property), $∃k\in Z^{+}\ni \frac{1}{k}<ε⟹∀n>k⟹\frac{1}{n}<\frac{1}{k}⟹\frac{1}{n}<ε$, so $\left|\frac{1}{n}-0\right|=\frac{1}{n}<ε ∀n>k$.

1. Show that $\{n\}$ be a divergent.

**Solution:** since if we assume that $\{n\}$ be a convergent $⟹∃x\in R\ni x\_{n}\rightarrow x$ and then $∀ε>0⟹(x-ε,x+ε)$ contains of terms $\{n\}$, since $x+ε\in R⟹$ (by Archimedes property) $⟹∃k\in Z^{+}\ni x+ε<k$, since $x+ε<k<k+1<…⟹k, k+1, k+2 \notin (x-ε,x+ε)$, this means $(x-ε,x+ε)$ does not contain on terms of $\{n\}$, but this is contradiction.

1. Show that $\{x\_{n}\}$ such that $x\_{n}=\left\{\begin{array}{c}n, n\leq 10^{6}\\1, n>10^{8}\end{array}\right.$ be a convergent and converges to one .

**Solution:** since $∀ε>0$, take $k>10^{6} ∀n>k⟹x\_{n}=1$ and then $\left|x\_{n}-1\right|=0<ε⟹x\_{n}\rightarrow 1$.

1. Show that $\{(-1)^{n}\}$ be a divergent.

**Solution:** since if we suppose that $\{(-1)^{n}\}$ be a convergent $⟹∃x\in R\ni x\_{n}=(-1)^{n}\rightarrow x$, let $ε>0 ∃k\in Z^{+}\ni \left|x\_{n}-x\right|<ε ∀n>k⟹\left|(-1)^{n}-x\right|<ε ∀n>k⟹x-ε<(-1)^{n}<x+ε ∀n>k⟹(-1)^{n}\in \left(x-ε,x+ε\right)∀n>k$.

Let $x=1$, take $ε=\frac{1}{4}⟹\frac{1}{4}ε>0⟹(-1)^{n}\in (1-\frac{1}{4}ε,1+\frac{1}{4}ε)∀n $is an even, $(-1)^{n}\notin (1-\frac{1}{4}ε,1+\frac{1}{4}ε)∀n $is an odd, this means that $(1-\frac{1}{4}ε,1+\frac{1}{4}ε)$ does not contain all terms of $\{(-1)^{n}\}$ and then $\{(-1)^{n}\}$ does not converge to $1$.

By same way we prove that $\{(-1)^{n}\}$ does not converge to $-1$.

Now, let $x\ne 1, x\ne -1$, let $a\_{1}=\left|1-x\right|,a\_{2}=\left|-1-x\right|$, take $ε\ni ε<a\_{1}, ε<a\_{2}$, we deduce that $\left(x-ε,x+ε\right)$ does not contain on any term of$\{(-1)^{n}\}⟹(-1)^{n}$ does not converge to $x$.

(3.20) **Theorem:**

1. If a real sequence is a convergent, then a converge point is a unique.
2. Every convergent sequence be Cauchy sequence.

**Proof:** (1) Let $ x\_{n}\rightarrow x, y\_{n}\rightarrow y\ni x\ne y$ and let $\left|x-y\right|=ε⟹ε>0$, since $ x\_{n}\rightarrow x⟹∃k\_{1}\in Z^{+}\ni \left|x\_{n}-x\right|<\frac{ε}{2}∀n>k\_{1}, x\_{n}\rightarrow y⟹∃k\_{2}\in Z^{+}\ni \left|x\_{n}-y\right|<\frac{ε}{2}∀n>k\_{2}$ put $k=$max $\left\{k\_{1},k\_{2}\right\}⟹\left|x\_{n}-x\right|<\frac{ε}{2}, \left|x\_{n}-y\right|<\frac{ε}{2} ∀n>k⟹ε=\left|x-y\right|=\left|\left(x\_{n}-x\right)+(x\_{n}-y\right|\leq \left|x\_{n}-x\right|+\left|x\_{n}-y\right|<\frac{ε}{2}+\frac{ε}{2}=ε$, but this is a contradiction $⟹x=y$.

(2) let $ \{x\_{n}\}$ be a convergent sequence $⟹∃x\in X\ni x\_{n}\rightarrow x$, let $ε>0$, since $ x\_{n}\rightarrow x⟹∃k\in Z^{+}\ni \left|x\_{n}-x\right|<\frac{ε}{2}∀n>k$, if $n,m\geq k⟹\left|x\_{n}-x\_{m}\right|=\left|\left(x\_{n}-x\right)+(x-x\_{m}\right|\leq \left|x\_{n}-x\right|+\left|x\_{m}-x\right|<\frac{ε}{2}+\frac{ε}{2}=ε$ and then $ \{x\_{n}\}$ be Cauchy sequence.

(3.21) **Definition**: If $ \{x\_{n}\}$ be a real sequence, we say that $ \{x\_{n}\}$ is

1. Bounded above, if $∃M\_{1}\in R\ni x\_{n}\leq M\_{1}∀n$;
2. Bounded below, if $∃M\_{2}\in R\ni M\_{2}\leq x\_{n}∀n$;
3. Bounded, if $∃M\in R\ni \left|x\_{n}\right|\leq M ∀n$.

(3.22) **Examples**:

1. $\{\frac{1}{n}\}$ is a bounded, since $\left|\frac{1}{2}\right|<2 ∀n$.
2. $\{\frac{n}{n+1}\}$ is a bounded, since $\left|\frac{n}{n+1}\right|<1 ∀n$.
3. $\{(-1)^{n}\}$ is a bounded, since $\left|(-1)^{n}\right|\leq 1 ∀n$.
4. $\{n\}$ does not bounded, since if we suppose that $\{n\}$ is a bounded $⟹∃M\in R^{+}\ni \left|n\right|\leq M ∀n$, but this is a contradiction (Archimedes property) since $n\geq M, n\in Z^{+}$.
5. $\{3^{n}\}$ does not bounded.

(3.23) **Theorem**: Every Cauchy sequence be a bounded, and then every convergent sequences be a bounded.

**Proof:** Let $\{x\_{n}\}$ be Cauchy sequence, we must prove that$ \{x\_{n}\}$ is a bounded. Let $ε=1$, since $\{x\_{n}\}$ is a Cauchy sequence $⟹∃k\in Z^{+}\ni \left|x\_{n}-x\_{m}\right|<1∀n,m>k$, let $m=k+1⟹\left|x\_{n}-x\_{m}\right|<1∀n>k$, since$ \left|x\_{n}\right|-\left|x\_{k+1}\right|\leq \left|x\_{n}-x\_{k+1}\right|⟹\left|x\_{n}\right|-\left|x\_{k+1}\right|<1 ∀n>k⟹\left|x\_{n}\right|<1+\left|x\_{k+1}\right| ∀n>k$, put $M=$ max$\{\left|x\_{1}\right|,\left|x\_{2}\right|,…,\left|x\_{k}\right|,\left|x\_{k+1}\right|+1\}$, and then $\{x\_{n}\}$ is a bounded.

(3.24) **Note**: If a real sequence is a bounded, then its not a necessary be a convergent, for example $\{(-1)^{n}\}$ is a bounded, but does not convergent.

(3.25) **Definition**: Let $\{x\_{n}\}$ be a real sequence. We said that $\{x\_{n}\}$

1. Non-decreasing, if $x\_{n}\leq x\_{n+1} ∀n$.
2. Increasing, if $x\_{n}<x\_{n+1} ∀n$.
3. Non-increasing, if $x\_{n+1}\leq x\_{n} ∀n$.
4. Decreasing, if $x\_{n+1}<x\_{n} ∀n$.

(3.26) **Note**: We said that $\{x\_{n}\}$ is a monotonic, if its be satisfy any one of above.

(3.26) **Examples**:

1. {$\frac{1}{\sqrt{2}}\}$ is a decreasing $⟹$ a monotonic.
2. $\{\frac{n}{n+1}\}$ is an increasing $⟹$ a monotonic.
3. $\{(-1)^{n}\}$ does not a monotonic.

(3.27) **Theorem**:

1. Every bounded real sequence and monotonic be a convergent.
2. Every bounded real sequence contains on a convergent partial sequence.

(3.28) **Theorem**: (Some special sequences)

1. If $p>0⟹x\_{n}=\frac{1}{n^{p}}\rightarrow 0$.
2. If $p>0⟹x\_{n}=n^{p}\rightarrow 1$.
3. $x\_{n}=\sqrt[n]{n}\rightarrow 1$.
4. If $\left|a\right|<1⟹x\_{n}=a^{n} \rightarrow 0$.

**Proof:** (1) let $ε>0$, take $k>(\frac{1}{ε})^{1/p} ∀n>k⟹n>(\frac{1}{ε})^{1/p}⟹\frac{1}{n^{p}}<ε⟹\left|\frac{1}{n^{p}}-0\right|<ε$.

(2) a. if $p>1⟹\sqrt[n]{p}>1$, put $y\_{n}=\sqrt[n]{p}-1⟹y\_{n}>0⟹\sqrt[n]{p}=1+y\_{n}⟹(1+y\_{n})^{n}=1+ny\_{n}+\frac{n\left(n-1\right)}{2}y\_{n}^{2}+…+y\_{n}^{n}$, since $y\_{n}>0 ∀n⟹p\geq 1+ny\_{n}⟹\frac{p-1}{n}\geq y\_{n}⟹0<y\_{n}<\frac{p-1}{n}⟹y\_{n}\rightarrow 0⟹x\_{n}\rightarrow 0$.

b. if $p=1⟹\sqrt[n]{p}=1 ∀n⟹\sqrt[n]{p}\rightarrow 1$.

c. if $0<p<1⟹\frac{1}{p}>1$, put $λ=\frac{1}{p}⟹\sqrt[n]{p}=\frac{1}{\sqrt[n]{λ}}$ and $\sqrt[n]{λ}\rightarrow 1$, since $λ>0⟹\sqrt[n]{p}\rightarrow 1$.

(3) let $y\_{n}=\sqrt[n]{n}-1$, since $\sqrt[n]{n}>1∀n⟹y\_{n}>0 ∀n⟹\sqrt[n]{n}=1+y\_{n}⟹n=(1+y\_{n})^{n}=1+ny\_{n}+\frac{n\left(n-1\right)}{2}y\_{n}^{2}+…+y\_{n}^{n}⟹n>\frac{n\left(n-1\right)}{2}y\_{n}^{2}⟹y\_{n}^{2}<\frac{2}{n-1}⟹\left|y\_{n}\right|<\sqrt{\frac{2}{n-1}}⟹y\_{n}<\sqrt{\frac{2}{n-1}} ∀n\geq 2⟹y\_{n}\rightarrow 0⟹x\_{n}\rightarrow 0$.

(4) $∀ε>0 ∃k\in Z^{+}\ni \left|x\_{n}\right|<ε ∀n>k$.

a. if $a=0,k=0$.

b. if $a\ne 0⟹\frac{1}{\left|a\right|}$ exists, put $b=\left|a\right|-1⟹\frac{1}{\left|a\right|}=1+b$, since $\left|a\right|<1⟹\frac{1}{\left|a\right|}>1⟹b>0$, $\left|a^{n}\right|=\left|a\right|^{n}=\frac{1}{(1+b)^{n}}$ , $(1+b)^{n}=1+nb+…+b^{n}>nb ⟹\frac{1}{(1+b)^{n}}<\frac{1}{nb}⟹\left|a^{n}\right|<\frac{1}{nb} ∀n,$ put $k>\frac{1}{εb}∀n>k⟹n>\frac{1}{εb}⟹\frac{1}{nb}<ε⟹\left|a^{n}\right|<ε⟹x\_{n}=a^{n}\rightarrow 0$.

(3.29) **Theorem**: Let $\{x\_{n}\}, \{y\_{n}\} $be a real sequences such that $x\_{n}\rightarrow x$ and $y\_{n}\rightarrow y$, then

1. $x\_{n}+y\_{n}\rightarrow x+y$.
2. $λx\_{n}\rightarrow λx ∀λ\in R$.
3. $λ+x\_{n}\rightarrow λ+x ∀λ\in R$.
4. $x\_{n}y\_{n}\rightarrow xy$.
5. $\frac{x\_{n}}{y\_{n}}\rightarrow \frac{x}{y}$ where $y\_{n}\ne 0 ∀n$.
6. $\frac{1}{y\_{n}}\rightarrow \frac{1}{y}$ where $y\_{n}\ne 0 ∀n$.
7. $\left|x\_{n}\right|\rightarrow \left|x\right|$.
8. $\left|x\_{n}-y\_{n}\right|\rightarrow \left|x-y\right|$.
9. If $x\_{n}\leq y\_{n}⟹x\leq y ∀n$.

**Proof:** (1) let $ε>0$, since $x\_{n}\rightarrow x⟹∃k\_{1}\in Z^{+}\ni \left|x\_{n}-x\right|<\frac{ε}{2}∀n>k\_{1}$, since $y\_{n}\rightarrow y⟹∃k\_{2}\in Z^{+}\ni \left|y\_{n}-y\right|<\frac{ε}{2}∀n>k\_{2}$, put $k=$ max $\left\{k\_{1}, k\_{2}\right\}⟹\left|x\_{n}-x\right|<\frac{ε}{2},\left|y\_{n}-y\right|<\frac{ε}{2}∀n>k$, $\left|(x\_{n}-x)+(x\_{n}-y)\right|\leq \left|x\_{n}-x\right|+\left|x\_{n}-x\right|<\frac{ε}{2}+\frac{ε}{2}=ε$, so $x\_{n}+y\_{n}\rightarrow x+y$.

(3.30) **Theorem**:

1. For all real number, there is Cauchy sequence of rational numbers converge of them.
2. For all real number, there is Cauchy sequence of irrational numbers converge of them.
3. There is Cauchy sequence of rational numbers does not converge to any rational number.

 **Proof:** (1) let $r\in R$, since $r-\frac{1}{n}<r<r+\frac{1}{n}∀n\in Z^{+}⟹$ (by density of rational numbers) $⟹∀n\in Z^{+}∃r\_{n}\in Q\ni \left|r\_{n}-r\right|<\frac{1}{n} ∀n\in Z^{+}⟹r-\frac{1}{n} <r\_{n}<r+\frac{1}{n}∀n\in Z^{+}$, now, we must prove that $r\_{n}\rightarrow r$, let $ε>0⟹$ (by Archimedes property) $⟹∃k\in Z^{+}\ni \frac{1}{k}< ε ∀n>k⟹\frac{1}{n}<\frac{1}{k}⟹\left|r\_{n}-r\right|<\frac{1}{n}<\frac{1}{k}<ε⟹r\_{n}\rightarrow r$.

(3.31) **Definition**: We said that a space $X$ is a complete, if every Cauchy sequence in $X$ be a convergent in $X$.

(3.32) **Note**: $Q$ is an incomplete, while $R$ is a complete.