References: *An Introduction to Dynamic Meteorology*, Holton

MOMENTUM EQUATIONS

● **The momentum equations governing the ocean or atmosphere are**

$$\frac{\partial u}{\partial t} + \vec{V} \bullet \nabla u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_x$$

$$\frac{\partial v}{\partial t} + \vec{V} \bullet \nabla v = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_y$$

$$\frac{\partial w}{\partial t} + \vec{V} \bullet \nabla w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \quad .$$

● **The terms F_x, F_y, and F_z represent dissipative effects due to molecular effects (viscosity), and are often simply referred to as friction.**

● **The Boussinesq approximation**

  o **Let density and pressure be represented by a "base" or reference state plus a perturbation, so that**

$$p(x, y, z, t) = p_0(z) + \tilde{p}(x, y, z, t)$$
$$\rho(x, y, z, t) = \rho_0(z) + \tilde{\rho}(x, y, z, t)$$

  If we also assume that the base state is in hydrostatic balance, and assume that $\tilde{p}$ can be ignored everywhere except where it is multiplied by gravity, we get the Boussinesq approximation momentum equations

$$\frac{\partial u}{\partial t} + \vec{V} \bullet \nabla u = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} + fv + F_x$$

$$\frac{\partial v}{\partial t} + \vec{V} \bullet \nabla v = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y} - fu + F_y$$

$$\frac{\partial w}{\partial t} + \vec{V} \bullet \nabla w = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial z} + b + F_z \quad .$$

  where

$$b \equiv -\frac{\tilde{\rho}}{\rho_0}g$$

  and is called the *buoyancy*.

● **The Boussinesq approximation is valid only if compressibility effects are small.**

o **In the atmosphere this means the Boussinesq approximation can only be reasonably applied over relatively shallow layers, such as the planetary boundary layer.**

o **In the ocean the Boussinesq approximation can often be applied over the entire depth, since water is far less compressible than air.**

TURBULENCE

- **Meteorological and oceanographic variables fluctuate rapidly around some base value.**

- **Standard weather observations record averaged wind speed and direction, not instantaneous speed and direction.**

  o **In the U.S. the standard wind speed is a two-minute average for synoptic observations.**

- **The rapidly fluctuating part is the turbulence.**

- **We are also mainly interested in the time-averaged effects of the turbulence, not its instantaneous effects.**

- **In order to isolate the time-averaged effects of turbulence from the mean values of the variables themselves, we perform *Reynolds averaging*.**

REYNOLDS AVERAGING

- **First decompose all of the dependent variables into mean and perturbation quantities**

$$u(x, y, z, t) = \bar{u}(x, y, z, t) + u'(x, y, z, t)$$
$$v(x, y, z, t) = \bar{v}(x, y, z, t) + v'(x, y, z, t)$$
$$w(x, y, z, t) = \bar{w}(x, y, z, t) + w'(x, y, z, t)$$
$$\tilde{p}(x, y, z, t) = \bar{p}(x, y, z, t) + p'(x, y, z, t)$$
$$\tilde{\rho}(x, y, z, t) = \bar{\rho}(x, y, z, t) + \rho'(x, y, z, t)$$

  o **The mean quantities, symbolized with the over-bar symbol ($\overline{}$) are time averaged quantities given by**

$$\bar{F} = \frac{1}{\tau} \int_{t}^{t+\tau} F(x, y, z, \tilde{t}) d\tilde{t}$$

  where τ is the averaging interval.

- o **The mean quantities are not necessarily constant in time or space.**
- o **The perturbation quantities are symbolized with a prime (´).**
- **After decomposing the dependent variables into mean and perturbation quantities we again time average the equations.**
- **The rules for the time-averaging operator:**
  - o **The sum of two means is the mean of the sum:** $\overline{A+B} = \overline{A} + \overline{B}$
  - o **The mean of a mean is just the mean (very Zen-like):** $\overline{\overline{A}} = \overline{A}$
  - o **The mean of a single perturbation is zero:** $\overline{A'} = 0$
  - o **The mean of the product of two means is just the product of the two means:**
    $$\overline{\overline{A}\,\overline{B}} = \overline{A}\,\overline{B}$$
  - o **The mean of the product containing a single perturbation is zero:** $\overline{A'\overline{B}} = 0$
  - o **The mean of the product of two perturbations is NOT zero:** $\overline{A'B'} \neq 0$
- **Example of Reynolds averaging the total derivative**
$$u = \overline{u} + u'$$
$$\frac{\partial(\overline{u}+u')}{\partial t} + (\overline{u}+u')\frac{\partial}{\partial x}(\overline{u}+u') + (\overline{v}+v')\frac{\partial}{\partial y}(\overline{u}+u') + (\overline{w}+w')\frac{\partial}{\partial z}(\overline{u}+u')$$

  which, when applying or rules for averaging, becomes
$$\frac{\partial\overline{u}}{\partial t} + \overline{u}\frac{\partial\overline{u}}{\partial x} + \overline{v}\frac{\partial\overline{u}}{\partial y} + \overline{w}\frac{\partial\overline{u}}{\partial z} + \overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}}.$$

  For an incompressible fluid it turns out that
$$u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial w'}{\partial z} = \frac{\partial}{\partial x}(u'u') + \frac{\partial}{\partial y}(v'u') + \frac{\partial}{\partial z}(w'u')$$

  so that the total derivative in the *u*-momentum equation is
$$\frac{\partial\overline{u}}{\partial t} + \overline{u}\frac{\partial\overline{u}}{\partial x} + \overline{v}\frac{\partial\overline{u}}{\partial y} + \overline{w}\frac{\partial\overline{u}}{\partial z} + \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{v'u'} + \frac{\partial}{\partial z}\overline{w'u'}.$$

- **The Reynolds averaged momentum equations are**

$$\frac{\bar{D}\bar{u}}{Dt} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \left[\frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{v'u'} + \frac{\partial}{\partial z}\overline{w'u'}\right]$$

$$\frac{\bar{D}\bar{v}}{Dt} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y} - f\bar{u} - \left[\frac{\partial}{\partial x}\overline{u'v'} + \frac{\partial}{\partial y}\overline{v'v'} + \frac{\partial}{\partial z}\overline{w'v'}\right]$$

$$\frac{\bar{D}\bar{w}}{Dt} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial z} - g\frac{\bar{\rho}}{\rho_0} - \left[\frac{\partial}{\partial x}\overline{u'w'} + \frac{\partial}{\partial y}\overline{v'w'} + \frac{\partial}{\partial z}\overline{w'w'}\right]$$

where

$$\frac{\bar{D}}{Dt} \equiv \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z}.$$

- **The over-barred terms are *turbulent momentum fluxes*.**
- **In most applications we can usually ignore these terms because they are small compared to the other terms in the equations. However, *in the atmospheric and oceanic boundary layers the turbulent momentum fluxes are important*.**
- **The thermodynamic energy equation for the atmosphere is**

$$\frac{DT}{Dt} + \frac{g}{c_p}w = J$$

  where *J* is the heating rate due to diabatic processes (latent heat release, radiation, etc.)

  o **The Reynolds averaged thermodynamic energy equation is**

$$\frac{\bar{D}\bar{T}}{Dt} + \frac{g}{c_p}\bar{w} + \frac{\partial}{\partial x}\overline{u'T'} + \frac{\partial}{\partial y}\overline{v'T'} + \frac{\partial}{\partial z}\overline{w'T'} = \bar{J}.$$

  o **The over-barred terms are *turbulent heat fluxes*.**
- **The conservation equation for water vapor is**

$$\frac{D\rho_v}{Dt} = P - L,$$

  where ρ_v is the absolute humidity (mass of water vapor per volume of air), *P* represents processes which produce water vapor (e.g., evaporation and sublimation), and *L* represents loss processes (e.g., condensation and deposition).

  o **The Reynolds averaged equation for water vapor is**

$$\frac{\bar{D}\bar{\rho}_v}{Dt} + \frac{\partial}{\partial x}\overline{u'\rho_v'} + \frac{\partial}{\partial y}\overline{v'\rho_v'} + \frac{\partial}{\partial z}\overline{w'\rho_v'} = \bar{P} - \bar{L}.$$

- o **The overbarred terms are *turbulent moisture fluxes*.**
- o **Moisture flux is sometimes called *latent heat flux*, because as the moisture condenses it releases heat.**
- **For air/sea interaction we are most concerned with the vertical fluxes (those with the w'.) We will usually ignore the horizontal fluxes.**

MORE ON FLUXES

- **A *dynamic flux* is defined as the flow of a quantity per unit area per unit time.**
- **A *kinematic flux* is defined as the dynamic flux divided by the density.**
- **To show that a term like $w'u'$ is a kinematic momentum flux, we look at the units.**
  - o $w'u'$ **has units of m^2/s^2.**
  - o **Kinematic momentum flux has units of**

$$\frac{(momentum)}{(area)(time)(density)} = \frac{(kg \cdot m / s)}{(m^2)(s)(kg/m^3)} = \frac{m^2}{s^2}.$$

REYNOLDS STRESSES

- **A *stress* is a force per unit area.**
  - o **Stresses can either be normal or tangential.**
  - o **Stress has same units as pressure.**
- **The turbulent momentum fluxes are also known as *Reynolds stresses*, because their effects on the shape of a volume of fluid are identical to that from a viscous stress.**
- **The relation between the momentum flux and Reynolds stress is**
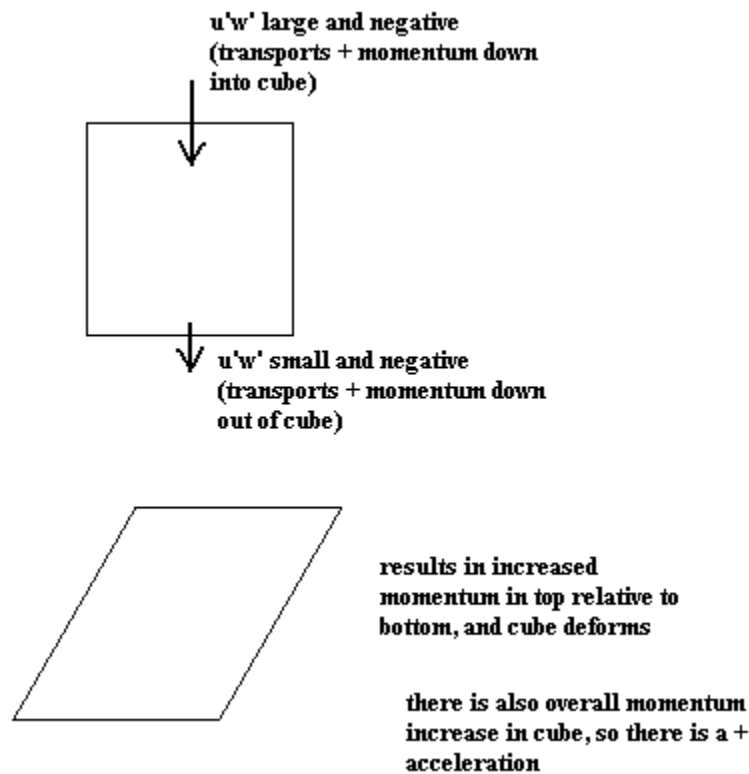
$$\tau_R / \rho_0 = -\overline{u'w'}.$$

and so the Reynolds averaged momentum equations can be written as

$$\frac{\overline{D}\overline{u}}{Dt} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\overline{v} + \frac{1}{\rho_0}\frac{\partial \tau_R}{\partial z}.$$

- **That the Reynolds stress terms act to deform a cube of fluid is illustrated for westerly flow ($\overline{u} > 0$) as follows:**

o **For westerly flow** $\overline{u'w'}$ **is usually negative, signifying a downward transport of positive momentum.**

o **If** $\overline{u'w'}$ **decreases with height (larger magnitude, but more negative) there is more positive momentum put into the top of the cube than is removed from it at the bottom (see diagram) and the cube will deform just as if it were under a stress.**

o **Also, since there is a net increase in momentum over the cube, the cube will accelerate. This is consistent with the equation**

$$\frac{D\overline{u}}{Dt} = -\frac{\partial}{\partial z}\overline{u'w'} > 0.$$

u'w' large and negative
(transports + momentum down
into cube)

results in increased
momentum in top relative to
bottom, and cube deforms

u'w' small and negative
(transports + momentum down
out of cube)

there is also overall momentum
increase in cube, so there is a +
acceleration

FLUXES AND AIR/SEA INTERACTION

● **The atmosphere interacts with the surface of the Earth. The effects of this interaction are to exchange momentum, heat, and moisture.**

● **This exchange is quantified in terms of the fluxes of momentum, moisture, and heat.**

- **Thus,** *understanding the vertical fluxes of heat, moisture, and momentum is key to understanding air-sea interaction.*